

Mathematics

FINITE ELEMENT ANALYSIS AS COMPUTATION

What the textbooks don't teach you about finite element analysis

Chapter 5: The locking phenomena

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Chapter 5

The locking phenomena

5.1 Introduction

A pivotal area in finite element methodology is the design of robust and accurate elements for applications in general purpose packages (GPPs) in structural analysis. The displacement type approach we have been examining so far is the technique that is overwhelmingly favored in general purpose software. We have seen earlier that the finite element method can be interpreted as a piecewise variation of the Rayleigh-Ritz method and therefore that it seeks strains/stresses in a 'best-fit' manner. From such an interpretation, it is possible to argue that errors, whether in displacements, stresses or energies, due to finite element discretization must converge rapidly, at least in a $(1/L)^2$ manner or better, where a large structure (domain) of dimension L is sub-divided into elements (sub-domains) of dimension l . Thus, with ten elements in a one-dimensional structure, errors must not be more than a few percent.

By and large, the elements work without difficulty. However, there were spectacular failures as well. These are what are now called the 'locking' problems in C^0 finite elements - as the element libraries of GPPs stabilized these elements came to be favored for reasons we shall discuss shortly. By locking, we mean that finite element solutions vanish quickly to zero (errors saturating quickly to nearly 100%!) as certain parameters (the penalty multipliers) become very large. It was not clear why the displacement type method, as it was understood around 1977, should produce for such problems, answers that were only a fraction of a per cent of the correct answer with a practical level of discretization. Studies in recent years have established that an aspect known as *consistency* must be taken into account.

The consistency paradigm requires that the interpolation functions chosen to initiate the discretization process must also ensure that any special constraints that are anticipated must be allowed for in a consistent way. Failure to do so causes solutions to lock to erroneous answers. The paradigm showed how elements can be designed to be free of these errors. It also enabled error-analysis procedures that allowed errors to be traced to the inconsistencies in the representation to be developed. We can now develop a family of such error-free robust elements for applications in structural mechanics.

In this chapter, we shall first introduce the basic concepts needed to understand why such discretized descriptions fail while others succeed. We compare the equations of the Timoshenko beam theory (a C^0 theory) to the classical beam theory (a C^1 theory) to show how constraints are generated in such a model. This permits us to discuss the concept of consistency and the nature of errors that appear during a Ritz type approximation. These same errors are responsible for the locking seen in the displacement type finite element models of similar problems.

5.2 From C^1 to C^0 elements

As the second generation of GPPs started evolving around the late 70s and early 80s, their element libraries replaced what were called the C^1 elements with what were known as the C^0 elements. The former were based on well known classical theories of beams, plates and shells, reflecting the confidence structural analysts had in such theories for over two centuries - namely the Euler-Bernoulli beam theory, the Kirchhoff-Love plate theory and the equivalent shell theories. These theories did not allow for transverse shear strain and

permitted the modeling of such structures by defining deformation in terms of a single field, w , the transverse deflection of a point on what is called the neutral axis (in a beam) and neutral surface of a plate or shell. The strains could then be computed quite simply from the assumption that normal to the neutral surface remained normal after deformation. One single governing differential equation resulted, although of a higher order (in comparison to other theories we shall discuss shortly), and this was considered to be an advantage.

There were some consequences arising from such an assumption both for the mathematical modeling aspect as well as for the finite element (discretization) aspect. In the former, it turned out that to capture the physics of deformation of thick or moderately thick structures, or the behavior of plates and shells made of newly emerging materials such as high performance laminated composites, it was necessary to turn to more general theories accounting for transverse shear deformation as well - these required the definition of rotations of normals which were different from the slopes of the neutral surface. Some of the contradictions that arose as a result of the old C^1 theories - e.g. the use of the fiction of the Kirchhoff effective shear reactions, could now be removed, restoring the more physically meaningful set of three boundary conditions on the edge of a plate or shell (the Poisson boundary conditions as they are called) to be used. The orders of the governing equations were correspondingly reduced. A salutary effect that carried over to finite element modeling was that the elements could be designed to have nodal degrees of freedom which were the six basic engineering degrees of freedom - the three translations and the three rotations at a point. This was ideal from the point of view of the organization of a general purpose package. Also, elements needed only simple basis functions requiring only the continuity of the fields across element boundaries - these are called the C^0 requirements. In the older C^1 formulations, continuity of slope was also required and to achieve this in arbitrarily oriented edges, as would be found in triangular or quadrilateral planforms of a plate bending element, it was necessary to retain curvature degrees of freedom ($w_{,xx}$, $w_{,xy}$, $w_{,yy}$) at the nodes and rely on quintic polynomials for the element shape or basis functions. So, as general purpose packages ideal for production run analyses and design increasingly found favour in industry, the C^0 beam, plate and shell elements slowly began to replace the older C^1 equivalents. It may be instructive to note that the general two-dimensional (i.e. plane stress, plane strain and axisymmetric) elements and three-dimensional (solid or brick as they are called) elements were in any case based on C^0 shape functions - thus this development was welcome in that universally valid C^0 shape functions and their derivatives could be used for a very wide range of structural applications.

However, life was not very simple - surprisingly dramatic failures came to be noticed and the greater part of academic activity in the late seventies, most of the eighties and even in the nineties was spent in understanding and eliminating what were called the locking problems.

5.3 Locking, rank and singularity of penalty-linked stiffness matrix, and consistency of strain-field

When locking was first encountered, efforts were made to associate it with the rank or non-singularity of the stiffness matrix linked to the penalty term (e.g. the shear stiffness matrix in a Timoshenko beam element which becomes very large as the beam becomes very thin, see the discussion below). However, on reflection, it is obvious that these are symptoms of the problem and not the cause. The high rank and non-singularity is the outcome of certain assumptions made (or not made, i.e. leaving certain unanticipated requirements unsatisfied) during the discretization process. It is therefore necessary to trace this to the origin. The

consistency approach argues that it is necessary in such problems to discretize the penalty-linked strain fields in a consistent way so that only physically meaningful constraints appear.

In this section, we would not enter into a formal finite element discretization (which would be taken up in the next section) but instead, illustrate the concepts involved using a simple Ritz-type variational method of approximation of the beam problem *via* both classical and Timoshenko beam theory [5.1]. It is possible to show how the Timoshenko beam theory can be reduced to the classical thin beam theory by using a penalty function interpretation and in doing so, show how the Ritz approximate solution is very sensitive to the way in which its terms are chosen. An 'inconsistent' choice of parameters in a low order approximation leads to a full-rank (non-singular) penalty stiffness matrix that causes the approximate solution to lock. By making it 'consistent', locking can be eliminated. In higher order approximations, 'inconsistency' does not lead to locked solutions but instead, produces poorer convergence than would otherwise be expected of the higher order of approximation involved. It is again demonstrated that a Ritz approximation that ensures *ab initio* consistent definition will produce the expected rate of convergence - a simple example will illustrate this.

5.3.1 The classical beam theory

Consider the transverse deflection of a thin cantilever beam of length L under an uniformly distributed transverse load of intensity q per unit length of the beam. This should produce a linear shear force distribution increasing from 0 at the free end to qL at the fixed end and correspondingly, a bending moment that varies from 0 to $qL^2/2$. Using what is called the classical or Euler-Bernoulli theory, we can state this problem in a weak form with a quadratic functional given by,

$$\Pi = \int_0^L \left(\frac{1}{2} EI w_{,xx}^2 - qw \right) dx \quad (5.1)$$

This theory presupposes a constraint condition, assuming zero transverse shear strain, and this allows the deformation of a beam to be described entirely in terms of a single dependent variable, the transverse deflection w of points on the neutral axis of the beam. An application of the principle of minimum total potential allows the governing differential equations and boundary conditions to be recovered, but this will not be entered into here. A simple exercise will establish that the exact solution satisfying the governing differential equations and boundary conditions is,

$$w(x) = \left(qL^2 / 4EI \right) x^2 - (qL/6EI) x^3 - (q/24EI) x^4 \quad (5.2a)$$

$$EI w_{,xx} = (q/2) (L - x)^2 \quad (5.2b)$$

$$EI w_{,xxx} = -q(L - x) \quad (5.2c)$$

5.3.1.1 A two-term Ritz approximation

Let us now consider an approximate Ritz solution based on two terms, $\bar{w} = b_2 x^2 + b_3 x^3$. Note that the constant and linear terms are dropped, anticipating the boundary conditions at the fixed end. One can easily work out that the approximate solution will emerge as,

$$\bar{w}(x) = \left(5qL^2/24EI\right)x^2 - (qL/12EI)x^3 \quad (5.3a)$$

so that the approximate bending moment and shear force determined in this Ritz process are,

$$EI \bar{w}_{,xx} = \left(5qL^2/12\right) - (qL/2)x \quad (5.3b)$$

$$EI \bar{w}_{,xxx} = -qL/2 \quad (5.3c)$$

If the expressions in Equations (5.2) and (5.3) are written in terms of the natural co-ordinate system ξ , where $x = (1 + \xi)L/2$ so that the ends of the beam are represented by $\xi = -1$ and $+1$, the exact and approximate solutions can be expanded as,

$$EI w_{,xx} = \left(qL^2/8\right) \left(4/3 - 2\xi - 1/3 (1 - 3\xi^2)\right) \quad (5.4)$$

$$EI \bar{w}_{,xx} = \left(qL^2/8\right) (4/3 - 2\xi) \quad (5.5)$$

The approximate solution for the bending moment is seen to be a 'best-fit' or 'least-squares fit' of the exact solution, with the points $\xi = \pm 1/\sqrt{3}$, which are the points where the second order Legendre polynomial vanishes, emerging as points where the approximate solution coincides with the exact solution. From Equations (5.2c) and (5.3c), we see that the shear force predicted by the Ritz approximation is a 'best-fit' of the actual shear force variation. Once again we confirm that the Ritz method seeks the 'best-approximation' of the actual state of stress in the region being studied.

5.3.2 The Timoshenko beam theory and its penalty function form

The Timoshenko beam theory [5.1] offers a physically more general formulation of beam flexure by taking into account the transverse shear deformation. The description of beam behavior is improved by introducing two quantities at each point on the neutral axis, the transverse deflection w and the face rotation θ so that the shear strain at each point is given by $\gamma = \theta - w_{,x}$, the difference between the face rotation and the slope of the neutral axis.

The total strain energy functional is now constructed from the two independent functions for $w(x)$ and $\theta(x)$, and it will now account for the bending (flexural) energy and an energy of shear deformation.

$$\Pi = \int_0^L \left(1/2 EI \theta_{,x}^2 + 1/2 \alpha (\theta - w_{,x})^2 - q w \right) dx \quad (5.6)$$

where the curvature $\kappa = \theta_{,x}$, the shear strain $\gamma = \theta - w_{,x}$ and $\alpha = kGA$ is the shear rigidity. The factor k accounts for the averaged correction made for the shear strain distribution through the thickness. For example, for a beam of rectangular cross-section, this is usually taken as $5/6$.

The Timoshenko beam theory will asymptotically recover the elementary beam theory as the beam becomes very thin, or as the shear rigidity becomes very large, i.e. $\alpha \rightarrow \infty$. This requires that the Kirchhoff constraint $\theta - w_{,x} \rightarrow 0$ must emerge in the limit. For a very large α , these equations lead directly to the simple fourth order differential equation for w of elementary

beam theory. Thus, this is secured very easily in the infinitesimal theory but it is this very same point that poses difficulties when a simple Ritz type approximation is made.

5.3.2.1 A two-term Ritz approximation

Consider now a two term Ritz approximation based on $\bar{\theta} = a_1 x$ and $\bar{w} = b_1 x$. This permits a constant bending strain (moment) approximation to be made. The shear strain is now given by

$$\bar{\gamma} = a_1 x - b_1 \quad (5.7)$$

and it would seem that this can represent a linearly varying shear force. The Ritz variational procedure now leads to the following set of equations,

$$\left\{ EI \begin{bmatrix} L & \theta \\ \theta & \theta \end{bmatrix} + \alpha \begin{bmatrix} L^3/3 - L^2/2 & \\ -L^2/2 & L \end{bmatrix} \right\} \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \begin{Bmatrix} \theta \\ qL^2/2 \end{Bmatrix} \quad (5.8)$$

Solving, we get

$$a_1 = \frac{-3qL^2}{12EI + \alpha L^2}; b_1 = -\frac{qL}{2\alpha} - \frac{1.5qL^3}{12EI + \alpha L^2} \quad (5.9)$$

As $\alpha \rightarrow \infty$, both a_1 and $b_1 \rightarrow 0$. This is a clear case of a solution 'locking'. This could have been anticipated from a careful examination of Equations (5.6) and (5.7). The penalty limit $\alpha \rightarrow \infty$ in Equation (5.6) introduces a penalty condition on the shear strain and this requires that the shear strain must vanish in the Ritz approximation process - from Equation (5.7), the conditions emerging are $a_1 \rightarrow 0$ and $b_1 \rightarrow 0$. Clearly, $a_1 \rightarrow 0$ imposes a zero bending strain $\theta_{,x} \rightarrow 0$ as well - this spurious constraint produces the locking action on the solution. Thus, a meaningful approximation can be made for a penalty function formulation only if the penalty linked approximation field is consistently modeled. We shall see how this is done next.

5.3.2.2 A three-term consistent Ritz approximation

Consider now a three term approximation chosen to satisfy what we shall call the consistency condition. Choose $\bar{\theta} = a_1 x$ and $\bar{w} = b_1 x + b_2 x^2$. Again, only a constant bending strain is permitted. But we now have shear strain of linear order as,

$$\bar{\gamma} = -b_1 + (a_1 - 2b_2)x \quad (5.10)$$

Note that as $\alpha \rightarrow \infty$, the constraint $a_1 - 2b_2 \rightarrow 0$ is consistently balanced and will not result in a spurious constraint on the bending field. The Ritz variational procedure leads to the following equation structure:

$$\left\{ EI \begin{bmatrix} L & \theta & \theta \\ \theta & \theta & \theta \\ \theta & \theta & \theta \end{bmatrix} + \alpha \begin{bmatrix} L^3/3 & -L^2/2 & -2L^3/3 \\ -L^2/2 & L & L^2 \\ -2L^3/3 & L^2 & 4L^3/3 \end{bmatrix} \right\} \begin{Bmatrix} a_1 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} \theta \\ qL^2/2 \\ qL^3/3 \end{Bmatrix} \quad (5.11)$$

It can be easily worked out from this that the approximate solution is given by,

$$\bar{\theta} = -\frac{qL^2}{6EI} x \quad (5.12a)$$

$$\bar{w} = \frac{qL}{\alpha} x - \frac{q}{2\alpha} x^2 + \frac{qL^2}{12EI} x^2 \quad (5.12b)$$

$$-EI \bar{\theta}_{,x} = qL^2/6 \quad (5.12c)$$

$$\alpha(\bar{\theta} - \bar{w}_{,x}) = q(L - x) \quad (5.12d)$$

There is no locking seen at all - the bending moment is now a least squares correct constant approximation of the exact quadratic variation (this can be seen by comparing Equation (5.12c) with Equations (5.4) and (5.5) earlier). The shear force is now correctly captured as a linear variation - the consistently represented field in Equation (5.12) being able to recover this even as a $\alpha \rightarrow \infty$!

A comparison of the penalty linked matrices in Equations (5.8) and (5.11) shows that while in the former, we have a non-singular matrix of rank 2, in the latter we have a singular matrix of rank 2 for a matrix of order 3. It is clear also that the singularity (or reduced rank) is a result of the consistent condition represented by $(\alpha_1 - 2b_2)$ in the linear part of the shear strain definition in Equation (5.10) - as a direct consequence, the third row of the penalty linked matrix in Equation (5.11) is exactly twice the first row. It is this aspect that led to a lot of speculation on the role the singularity or rank of the matrix plays in such problems. We can further show that non-singularity of penalty linked matrix arises in an inconsistent formulation only when the order of approximation is low, as seen for the two-term Ritz approximation. We can go for a quadratic inconsistent approximation (with linear bending strain variation) in the next section to show that there is no locking' of the solution and that the penalty linked matrix is not non-singular - however the effect of inconsistency is to reduce the performance of the approximation to a sub-optimal level.

5.3.2.3 A four-term inconsistent Ritz approximation

We now take up a four term approximation which provides theoretically for a linear variation in the approximation for bending strain, i.e. $\bar{\theta} = a_1x + a_2x^2$ and $\bar{w} = b_1x + b_2x^2$ so that $\bar{\kappa} = \bar{\theta}_{,x} = a_1 + 2a_2x$ and the shear strain is

$$\bar{\gamma} = -b_1 + (a_1 - 2b_2)x + a_2x^2 \quad (5.13)$$

Note now that the condition $\alpha \rightarrow \infty$ forces $a_2 \rightarrow 0$ this becomes a spurious constraint on the bending strain field. We shall now see what effect this spurious constraint has on the approximation process. The Ritz variational procedure leads to the following set of equations:

$$\left\{ EI \begin{bmatrix} L & L^2 & 0 & 0 \\ L^2 & 4L^3/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \right. \quad (5.14)$$

$$\left. \alpha \begin{bmatrix} L^3/3 & L^4/4 & -L^2/2 & -2L^3/3 \\ L^4/4 & L^5/5 & -L^3/3 & -L^4/2 \\ -L^2/2 & -L^3/3 & L & L^2 \\ -2L^3/3 & -L^4/2 & L^2 & 4L^3/3 \end{bmatrix} \right\} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ qL^2/2 \\ qL^3/3 \end{Bmatrix}$$

It can be seen that the penalty linked 4×4 matrix is singular as the fourth row is exactly twice the second row - this arises from the consistent representation $(a_1 - 2b_2)$ of the linear part of shear strain in Equation (5.13). The rank of the matrix is therefore 3 and the solution should be free of "locking" - however the inconsistent constraint forces $a_2 \rightarrow 0$ and this means that the computed bending strain $\bar{\kappa} \rightarrow a_1$; i.e. it will have only a constant bending moment prediction capability. What it means is that this four term inconsistent approach will produce answers only as efficiently as the three term consistent Ritz formulation. Indeed, one can work out from Equation (5.14) that,

$$a_1 = -\frac{qL^2}{6EI} - \frac{15qL^2}{60EI + \alpha L^2} \quad (5.15a)$$

$$a_2 = \frac{15qL^2}{60EI + \alpha L^2} \quad (5.15b)$$

$$b_1 = -\frac{qL}{\alpha} - \frac{5qL^3}{2(60EI + \alpha L^2)} \quad (5.15c)$$

$$b_2 = -\frac{qL^2}{6EI} + \frac{q}{2\alpha} \quad (5.15d)$$

As $\alpha \rightarrow \infty$, the bending moment and shear force variation are given by,

$$-EI \bar{\theta}_{,x} = qL^2/6 \quad (5.16a)$$

$$\alpha(\bar{\theta} - \bar{w}_{,x}) = q(L - x) + 2.5qL(6(x/L)^2 - 6(x/L) + 1) \quad (5.16b)$$

The solution can sense only a constant bending moment in the thin beam limit. There are now violent quadratic oscillations in the shear force and these oscillations can be shown to vanish at the points $x/L = 1/2(1 + 1/\sqrt{3})$ and $1/2(1 - 1/\sqrt{3})$, or $\xi = \pm 1/\sqrt{3}$, which are the points corresponding to the 2 point Gauss-Legendre integration rule. The effect of the inconsistent representation has been to reduce the effectiveness of the approximation. We shall next see

how the effectiveness can be improved by making the approximation consistent before the variational process is carried out.

5.3.2.4 A four-term consistent Ritz approximation

Let us now take up a Ritz approximation with a consistently represented function for the shear strain defined as $\bar{\gamma}$. This can be achieved by noting that in Equation (5.13), the offending term is the quadratic term associated with a_2 . We also see from Equation (5.16b) that this leads to a spuriously excited quadratic form $(6x^2 - 6Lx + L^2)$. Our experience with consistent finite element formulations [5.2] allows us to replace the x^2 term in Equation (5.13) with $Lx - L^2/6$ so that,

$$\begin{aligned}\bar{\gamma} &= -b_1 + (a_1 - 2b_2)x + a_2(Lx - L^2/6) \\ &= (-b_1 - a_2L^2/6) + (a_1 + a_2L - 2b_2)x\end{aligned}\quad (5.17)$$

In fact, it can be proved using the generalized (mixed) variational theorem such as the Hellinger-Reissner and the Hu-Washizu principles [5.2] that the variationally correct way to determine $\bar{\gamma}$ from the inconsistent $\bar{\gamma}$ is,

$$\int \delta \bar{\gamma}^T (\bar{\gamma} - \bar{\gamma}) dx = 0$$

and this will yield precisely the form represented in Equation (5.17). The Ritz variational procedure now leads to,

$$\left\{ EI \begin{bmatrix} L & L^2 & 0 & 0 \\ L^2 & 4L^3/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \right. \quad (5.18)$$

$$\left. \alpha \begin{bmatrix} L^3/3 & L^4/4 & -L^2/2 & -2L^3/3 \\ L^4/4 & 7L^5/36 & -L^3/3 & -L^4/2 \\ -L^2/2 & -L^3/3 & L & L^2 \\ -2L^3/3 & -L^4/2 & L^2 & 4L^3/3 \end{bmatrix} \right\} \begin{Bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ qL^2/2 \\ qL^3/3 \end{Bmatrix}$$

It is important to recognize now that since the penalty linked matrix emerges from the terms $-(b_1 + a_2L^2/6)$ and $(a_1 + a_2L - 2b_2)$ there would only be two linearly independent rows and therefore the rank of the matrix is now 2. The approximate solution is then given by,

$$-EI \bar{\kappa} = \frac{5}{12} qL^2 - \frac{qL}{2} x \quad (5.19a)$$

$$\alpha \bar{\gamma} = q(L - x) \quad (5.19b)$$

Comparing Equation (5.19a) with Equation (5.3b) we see that the approximate solution to the Timoshenko equation for this problem with a consistent shear strain assumption gives exactly the same bending moment as the Ritz solution to the classical beam equation could provide. We also see that Equation (5.19b) is identical to Equation (5.12b) so that the shear force is now being exactly computed. In other words, as $\alpha \rightarrow \infty$, the terms a_1 , a_2 , b_1 and b_2 all yield physically meaningful conditions representing the state of equilibrium correctly.

5.4 Consistency and C^0 displacement type finite elements

We shall see in the chapters to follow that locking, poor convergence and violent stress oscillations seen in C^0 displacement type finite element formulations are due to a lack of consistent definition of the critical strain fields when the discretization is made - i.e. of the strain-fields that are constrained in the penalty regime.

The foregoing analysis showed how the lack of consistency translates into a non-singular matrix of full rank that causes locking in low-order Ritz approximations of such problems. It is also seen that in higher order approximations, the situation is not as dramatic as to be described as locking, but is damaging as to produce poor convergence and stress oscillations. It is easy to predict all this by examining the constrained strain-field terms from the consistency point of view rather than performing a post-mortem examination of the penalty-linked stiffness matrix from rank or singularity considerations as is mostly advocated in the literature.

We shall now attempt a very preliminary heuristic definition of these requirements as seen from the point of view of the developer of a finite element for an application in a constrained media problem. We shall see later that if a simple finite element model of the Timoshenko beam is made), the results are in very great error and that these errors grow without limit as the beam becomes very thin. This is so even when the shape functions for the w and θ have been chosen to satisfy the *completeness* and *continuity* conditions. We saw in our Ritz approximation of the Timoshenko beam theory in this section that as the beam becomes physically thin, the shear strains must vanish and it must begin to enforce the Kirchhoff constraint and that this is not possible unless the approximate field can correctly anticipate this. In the finite element statement of this problem, the shape functions chosen for the displacement fields cannot do this in a meaningful manner - spurious constraints are generated which cause locking. The consistency condition demands that the discretized strain field interpolations must be so constituted that it will enforce only physically true constraints when the discretized functionals for the strain energy of a finite element are constrained.

We can elaborate on this definition in the following descriptive way. In the development of a finite element, the field variables are interpolated using interpolations of a certain order. The number of constants used will depend on the number of nodal variables and any additional nodeless variables (those corresponding to bubble functions). From these definitions, one can compute the strain fields using the strain-displacement relations. These are obtained as interpolations associated with the constants that were introduced in the field variable interpolations. Depending on the order of the derivatives of each field variable appearing in the definition of that strain field (e.g. the shear strain in a Timoshenko theory will have θ and the first derivative of w), the coefficients of the strain field interpolations may have constants from all the contributing field variable interpolations or from only one or some of these. In some limiting cases of physical behavior, these strain fields can be constrained to be zero

values, e.g. the vanishing shear strain in a thin Timoshenko beam. Where the strain-field is such that all the terms in it (i.e. constant, linear, quadratic, etc.) have, associated with it, coefficients from all the independent interpolations of the field variables that appear in the definition of that strain-field, the constraint that appears in the limit can be correctly enforced. We shall call such a representation *field-consistent*. The constraints thus enforced are *true constraints*. Where the strain-field has coefficients in which the contributions from some of the field variables are absent, the constraints may incorrectly constrain some of these terms. This *field-inconsistent* formulation is said to enforce additional *spurious constraints*. For simple low order elements, these constraints are severe enough to produce solutions that rapidly vanish - causing what is often described as locking.

5.5 Concluding remarks

These exercises show us why it is important to maintain consistency of the basis functions chosen for terms in a functional, which are linked, to penalty multipliers. The same conditions are true for the various finite element formulations where locking, poor convergence and stress oscillations are known to appear. It is also clear why the imposition of the consistency condition into the formulation allows the correct rank or singularity of the penalty linked stiffness matrix to be maintained so that the system is free of locking or sub-optimal convergence. Again, it is worthwhile to observe that non-singularity of the penalty linked matrix occurs only when the approximate fields are of very low order as for the two-term Ritz approximation. In higher order inconsistent formulations, as for the four-term inconsistent Ritz approximation, solutions are obtained which are sub-optimal to solutions that are possible if the formulation is carried out with the consistency condition imposed *a priori*. We shall see later that the use of devices such as reduced integration permits the consistency requirement to be introduced when the penalty linked matrix is computed so that the correct rank which ensures the imposition of the true constraints only is maintained.

In the next chapter, we shall go to the locking and other over-stiffening phenomena found commonly in displacement finite elements. The phenomenon of shear locking is the most well known - we shall investigate this closely. The Timoshenko beam element allows the problem to be exposed and permits a mathematically rigorous error model to be devised. The *consistency* paradigm is introduced to show why it is essential to remove shear locking. The correctness concept is then brought in to ensure that the devices used to achieve a consistent strain interpolation are also variationally correct. This example serves as the simplest case that could be employed to demonstrate all the principles involved in the consistency paradigm. It sets the scene for the need for new paradigms (*consistency* and *correctness*) to complement the existing ones (*completeness* and *continuity*) so that the displacement model can be scientifically founded. The application of these concepts to Mindlin plate elements is also reviewed.

The *membrane locking* phenomenon is investigated in Chapter 7. The simple curved beams are used to introduce this topic. This is followed up with studies of *parasitic shear* and *incompressible locking*.

In Chapters 5 to 7 we investigate the effect of constraints on strains due to the physical regimes considered, e.g. vanishing shear strains in thin Timoshenko beams or Mindlin plates, vanishing membrane strains in curved beams and shells etc. In Chapter 8 we proceed to a few problems where consistency between strain and stress functions are needed even where no constraints on the strains appear. These examples show the universal need of the *consistency*

aspect and also the power of the general Hu-Washizu theorem to establish the complete, correct and consistent variational basis for the displacement type finite element method.

5.6 References

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