

Mathematics

FINITE ELEMENT ANALYSIS AS COMPUTATION

What the textbooks don't teach you about finite element analysis

Chapter 4: Convergence and Errors

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Chapter 4

Convergence and Errors

4.1 Introduction

Error analysis is that aspect of fem knowledge, which can be said to belong to the "second-order tradition" of knowledge. To understand its proper place, let us briefly review what has gone before in this text. We have seen structural engineering appear first as art and technology, starting as the imitation of structural form by human fabrication. The science of solid and structural mechanics codified this knowledge in more precise terms. Mathematical models were then created from this to describe the structural behavior of simple and complex systems analytically and quantitatively. With the emergence of cheap computational power, techniques for computational simulation emerged. The finite element method is one such device. It too grew first as art, by systematic practice and refinement. In epistemological terms, we can see this as a first-order tradition or level or inner loop of fem art and practice. Here, we know what to do and how to do it. However, the need for an outer loop or second-order tradition of enquiry becomes obvious. How do we know why we should do it this way? If the first stage is the stage of action, this second stage is now of reflection. Error analysis therefore belongs to this tradition of epistemological enquiry.

Let us now translate what we mean by error analysis into simpler terms in the present context. We must first understand that the fem starts with the basic premises of its paradigms, its definitions and its operational procedures to provide numerical results to physical problems which are already described by mathematical models that make analytical quantification possible. The answer may be wrong because the mathematical model wrongly described the physical model. We must take care to understand that this is not the issue here. If the mathematical model does not explain the actual physical system as observed through carefully designed empirical investigations, then the models must be refined or revised so that closer agreement with the experiment is obtained. This is one stage of the learning process and it is now assumed that over the last few centuries we have gone through this stage successfully enough to accept our mathematical models without further doubt or uncertainty.

Since our computational models are now created and manipulated using digital computers, there are errors which occur due to the fact that information in the form of numbers can be stored only to a finite precision (word length as it is called) at every stage of the computation. These are called round-off errors. We shall assume here that in most problems we deal with, word length is sufficient so that round-off error is not a major headache.

The real issue that is left for us to grapple with is that the computational model prepared to simulate the mathematical model may be faulty and can lead to errors. In the process of replacing the continuum region by finite elements, errors originate in many ways. From physical intuition, we can argue that this will depend on the type and shape of elements we use, the number of elements used and the grading or density of the mesh used, the way distributed loads are assigned to nodes, the manner in which boundary conditions are modeled by specification of nodal degrees of freedom, etc. These are the discretization errors that can occur.

Most of such errors are difficult to quantify analytically or determine in a logically coherent way. We can only rely on heuristic judgement to understand how best to minimize errors. However, we shall now look only at that category of discretization error that appears because

the computational or discretized model uses trial functions, which are an approximation of the true solution to the mathematical model. It seems possible that to some extent, analytical quantification of these errors is possible.

We can recognize two kinds of discretization error belonging to this category. The first kind is that which arises because a model replaces a problem with an infinitely large number of degrees of freedom with a finite number of degrees of freedom. Therefore, except in very rare cases, the governing differential equations and boundary conditions are satisfied only approximately. The second kind of error appears due to the fact that by overlooking certain essential requirements beyond that specified by continuity and completeness, the mathematical model can alter the physics of the problem. In both cases, we must be able to satisfy ourselves that the discretization process, which led to the computational model, has introduced a certain predictable degree of error and that this converges at a predictable rate, i.e. the error is removed in a predictable manner as the discretization is improved in terms of mesh refinement. Error analysis is the attempt to make such predictions *a priori*, or rationalize the errors in a logical way, *a posteriori*, after the errors are found.

To carry out error analysis, new procedures have to be invented. These must be set apart from the first-order tradition procedures that carry out the discretization (creating the computational model from the mathematical model) and solution (computing the approximate results). Thus, we must design auxiliary procedures that can trace errors in an *a priori* fashion from basic paradigms (conjectures or guesses). These error estimates or predictions can be seen as consequences computed from our guesses about how the fem works. These errors must now be verified by constructing simple digital computation exercises. This is what we seek to do now. If this cycle can be completed, then we can assure ourselves that we have carved out a scientific basis for error analysis. This is a very crucial element of our study. The fem, or for that matter, any body of engineering knowledge, or engineering methodology, can be said to have acquired a scientific basis only when it has incorporated within itself, these auxiliary procedures that permit its own self-criticism. Therefore, error analysis, instead of being only *a posteriori* or *post mortem* studies, as it is usually practised, must ideally be founded on *a priori* projections computed from intelligent paradigms which can be verified (falsified) by digital computation.

In this chapter, we shall first take stock of the conventional wisdom regarding convergence. This is based on the old paradigm that fem seeks to approximate displacements accurately. We next take note of the newly established paradigm that the Ritz-type and fem approximations seek strains/stresses in a 'best-fit' manner. From such an interpretation, we examine if it is possible to argue that errors, whether in displacements, stresses or energies, due to finite element discretization must diminish rapidly, at least in a $(l/L)^2$ manner or better, where a large structure (domain) of dimension L is sub-divided into elements (sub-domains) of dimension l . Thus, with ten elements in a one-dimensional structure, errors must not be more than a few percent. This is the usual range of problems where the continuity and completeness paradigms explain completely the performance of finite elements. In subsequent chapters, we shall discover however that a class of problems exist where errors are much larger - the discretization fail in a dramatic fashion. Convergence and error analysis must now be founded on a more complex conceptual framework - new paradigms need to be introduced and falsified. This will be postponed to subsequent chapters.

4.2 Traditional order of error analysis

The order of error analysis approach that is traditionally used is inherited from finite difference approaches. This seems reasonable because quite often simple finite difference and finite element approximations result in identical equations. It is also possible with suitable interpretations to cast finite difference approximations as particular cases of weighted residual approximations using finite element type trial functions. However, there are some inherent limitations in this approach that will become clear later.

It is usual to describe the magnitude of error in terms of the mesh size. Thus, if a series of approximate solutions using grids whose mesh sizes are uniformly reduced is available, it may be possible to obtain more information about the exact answer by some form of extrapolation, provided there is some means to establish the rate at which errors are removed with mesh size.

The order of error analysis proceeds from the understanding that the finite element method seeks approximations for the displacement fields. The errors are therefore interpreted using Taylor Series expansions for the true displacement fields and truncations of these to represent the discretized fields. The simple example below will highlight the essential features of this approach.

4.2.1 Error analysis of the axially loaded bar problem

Let us now go back to the case of the cantilever bar subjected to a uniformly distributed axial load of intensity q_0 (Section 2.3). The equilibrium equation for this problem is

$$u_{,xx} + q_0/AE = 0 \quad (4.1)$$

We shall use the two-node linear bar elements to model this problem. We have seen that this leads to a solution, which gives exact displacements at the nodes. It was also clear to us that this did not mean that an exact solution had been obtained; in fact while the true solution required a quadratic variation of the displacement field $u(x)$, the finite element solution $\bar{u}(x)$ was piecewise linear. Thus, within each element, there is some error at locations between the nodes.

Fig. 4.1 shows the displacement and strain error in the region of an element of length $2l$ placed with its centroid at x_i in a cantilever bar of length L . If for convenience we choose $q_0/AE = 1$, then we can show that

$$u(x) = Lx - x^2/2 \quad (4.2a)$$

$$\bar{u}(x) = Lx - (l^2 + 2xx_i - x_i^2)/2 \quad (4.2b)$$

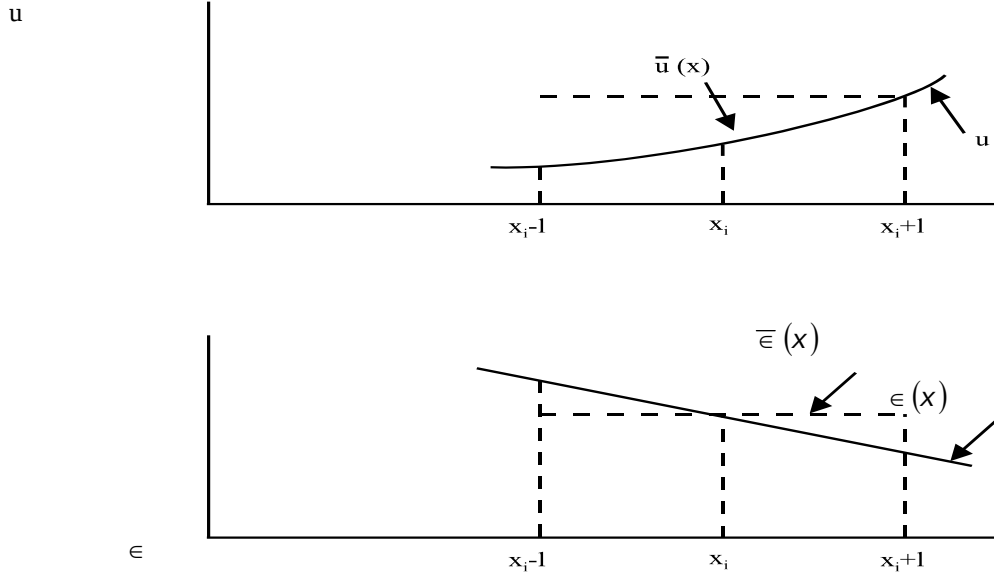


Fig. 4.1 Displacement and strain error in a uniform bar under distributed axial

load q_0

$$\varepsilon(x) = L - x \quad (4.2c)$$

$$\bar{\varepsilon}(x) = L - x_i \quad (4.2d)$$

If we denote the errors in the displacement field and strain field by $e(x)$ and $e'(x)$ respectively, we can show that

$$e(x) = (x - x_{i-1} + 1)(x - x_{i+1} - 1)/2 \quad (4.3a)$$

$$e'(x) = (x - x_i) \quad (4.3b)$$

From this, we can argue that in this case, the strain error vanishes at the element centroid, $x = x_i$, and that it is a maximum at the element nodes. This is clear from Fig. 4.1. It is also clear that for this problem, the displacement error is a maximum at $x = x_i$. What is more important to us is to derive measures for these errors in terms of the element size l , or more usefully, in terms of the parameter $h = l/L$, which is the dimensionless quantity that indicates the mesh division relative to the size of the structure. It will also be useful to have these errors normalized with respect to typical displacements and strains occurring in the problem. From Equations (4.2) and (4.3), we can estimate the maximum normalized errors to be of the following orders of magnitude, where “O” stands for “order”,

$$|e/u|_{\max} = O(h^2) \quad (4.4a)$$

$$|e'/\varepsilon|_{\max} = O(h) \quad (4.4b)$$

Note that the order of error we have estimated for the displacement field is that for a location inside the element as in this problem, the nodal deflections are exact. The strain error is $O(h)$ while the displacement error is $O(h^2)$. At element centroids, the error in strain vanishes. It is not clear to us in this analysis that the discretized strain $\bar{\varepsilon}(x)$ is a "best-fit" of the true strain $\varepsilon(x)$. If this is accepted, then it is possible to show that the error in the strain energy stored in such a situation is $O(h^2)$ as well.

It is possible to generalize these findings in an approximate or tentative way for more complex problems discretized using elements of greater precision. Thus, if an element with q nodes is used, the trial functions are of degree $q-1$. If the exact solution requires a polynomial field of degree q at least, then the computed displacement field will have an error $O(h^q)$. If strains for the problem are obtained as the r^{th} derivative of the displacement field, then the error in the strain or stress is $O(h^{q-r})$. Of course, these measures are tentative, for errors will also depend on how the loads are lumped at the nodes and so on.

4.3 Errors and convergence from "best-fit" paradigm

We have argued earlier that the finite element method actually proceeds to compute strains/stresses in a 'best-fit' manner within each element. We shall now use this argument to show how convergence and errors in a typical finite element model of a simple problem can be interpreted. We shall see that a much better insight into error and convergence analysis emerges if we base our treatment on the 'best-fit' paradigm.

We must now choose a suitable example to demonstrate in a very simple fashion how the convergence of the solution can be related to the best-fit paradigm. The bar under axial load, an example we have used quite frequently so far, is not suitable for this purpose. A case such as the uniform bar with linearly varying axial load modeled with two-node elements gives nodal deflections, which are exact, even though the true displacement field is cubic but between nodes, in each element, only a linear approximate trial function is possible. It can actually be proved that this arises from the special mathematical nature of this problem. We should therefore look for an example where nodal displacements are not exact.

The example that is ideally suited for this purpose is a uniform beam with a tip load as shown in Fig. 4.2. We shall model it with linear (two-node) Timoshenko beam elements which represent the bending moment within each element by a constant. Since the bending moment varies linearly over the beam for this problem, the finite element will replace this with a staircase approximation. Thus, with increase in number of elements, the stress pattern will approach the true solution more closely and therefore the computed strain energy due to bending will also converge. Since the applied load is at the tip, it is very easy to associate this load with the deflection under the load using Castigliano's theorem. It will then be possible to discern the convergence for the tip deflection. Our challenge is therefore to see if the best-fit paradigm can be used to predict the convergence rate for this example from first principles.

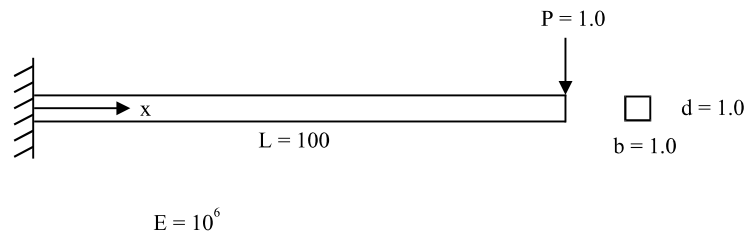


Fig. 4.2 Cantilever beam under tip load

4.3.1 Cantilever beam idealized with linear Timoshenko beam elements

The dimensions of a cantilever under tip load (Fig. 4.2) are chosen such that the tip deflection under the load will be $w=4.0$. The example chosen represents a thin beam so that the influence of shear deformation and shear strain energy is negligible.

We shall now discretize the beam using equal length linear elements based on Timoshenko theory. We use this element instead of the classical beam element for several reasons. This element serves to demonstrate the features of shear locking which arise from an inconsistent definition of the shear strains (It is therefore useful to take up this element for study later in Chapter 6). After correcting for the inconsistent shear strain, this element permits constant bending and shears strain accuracy within each element - the simplest representation possible under the circumstances and therefore an advantage in seeing how it works in this problem.

We shall restrict attention to the bending moment variation as we assume that the potential energy stored is mainly due to bending strain and that we can neglect the transverse shear strain energy for the dimensions chosen.

Figure 4.3 shows the bending moment diagrams for a 1, 2 and 4 element idealizations of the present problem using the linear element. The true bending moment (shown by the solid line) varies linearly. The computed (i.e. discretized) bending moments are distributed in a piecewise constant manner as shown by the broken lines. In each case, the elements pick up the bending moment at the centroid correctly - i.e. it is doing so in a 'best-fit' manner. What we shall now attempt to do for this problem is to relate this to the accuracy of results. We shall now interpret accuracy in the conventional sense, as that of the deflection at the tip under the load. Table 4.1 shows the

Table 4.1 Normalized tip deflections of a thin cantilever beam, $L/t = 100$

No. of elements	1	2	3
Predicted rate	.750	.938	.984
Element without RBF	.750	.938	.984
Element with RBF	1.000	1.000	1.000

normalized tip deflection with increasing idealizations (neglecting a very small amount due to shear deformation). An interesting pattern emerges. If error is measured by the norm $\{w - w(fem)\} / w$, it turns out that this is given exactly by the formula $1/4N^2$ where N is the number of elements. It can now be seen that this relationship can be established by arguing that this feature emerges from the fact that strains are sought in the 'best-fit' manner shown in Fig. 4.3.

Consider a beam element of length $2l$ (Fig. 4.4). Let the moment and shear force at the centroid be M and V . Thus the true bending moment over the element region for our problem can be taken to vary as $M + Vx$. The discretized bending moment sensed by our linear element would therefore be M . We shall now compute the actual bending energy in the element region (i.e. from a continuum analysis) and that given by the finite element (discretized) model. We can show that

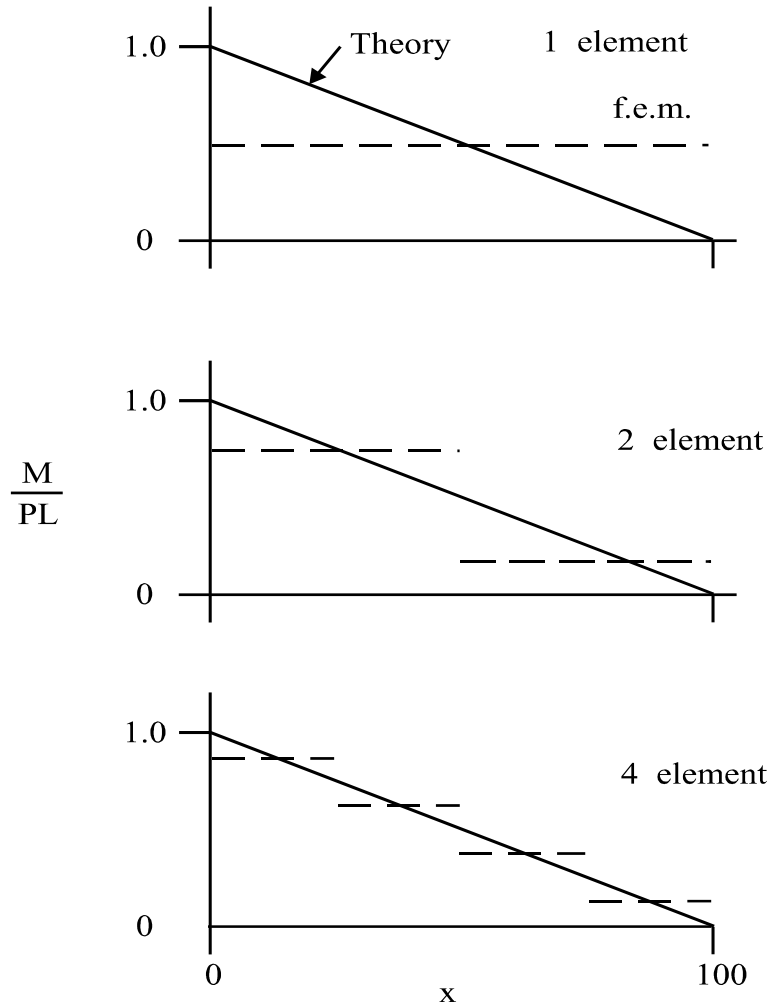


Fig. 4.3 Bending moment diagrams for a one-, two- and four-element dealizations of a cantilever beam under tip load.

$$\text{Energy in continuum model} = (1/EI) (M^2 + V^2 l^2 / 3) \quad (4.5)$$

$$\text{Energy in discretized model} = (1/EI) (M^2) \quad (4.6)$$

Thus, as a result of the discretization process involved in replacing each continuum segment of length $2l$ by a linear Timoshenko beam element which can give only a constant value M for the bending moment, there is a reduction (error) in energy in each element equal to $(1/EI) (V^2 l^2 / 3)$. It is simple now to show from this that for the cantilever beam of length L with a tip load P , the total reduction in strain energy of the discretized model for the beam is $U/4N^2$ where $U = P^2 L^3 / 6EI$ is the energy of the beam under tip load.

We are interested now to discover how this error in strain energy translates into an error in the deflection under load P . This can be very easily deduced using Castigliano's second theorem. It is left to the reader to show that the tip deflections of the continuum and discretized model will differ as $\{w - w(fem)\} / w = 1/4N^2$.

Table 4.1 shows this predicted rate of convergence. Our foregoing analysis shows that this follows from the fact that if any linear variation is approximated in a piecewise manner by constant values as seen in Fig. 4.3, this is the manner in which the square of the error in the stresses/strains (or, equivalently, the difference in work or energy) will converge with idealization. Of course, in a problem where the bending moment is constant, the rate of convergence will be better than this (in fact, exact) and in the case where the bending moment is varying quadratically or at a higher rate, the rate of convergence will be decidedly less.

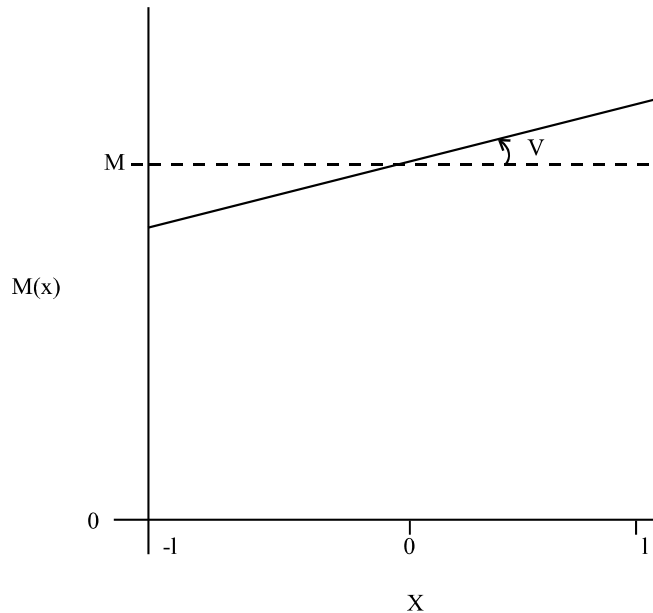


Fig. 4.4 Bending moment variation in a linear beam element.

We also notice that convergence in this instance is from below. This can be deduced from the fact that the discretized potential energy $\bar{U}$ is less than the actual potential energy U for this problem. It is frequently believed that the finite element displacement approach always underestimates the potential energy and a displacement solution is consequently described as a *lower bound solution*. However, this is not a universally valid generalization. We can see

briefly below (the reader is in fact encouraged to work out the case in detail) where the cantilever beam has a uniformly distributed load acting on it using the same linear Timoshenko beam element for discretization that this is not the case. It turns out that tip rotations converge from above (in a $1/2N^2$ rate) while the tip deflections are fortuitously exact. The *lower bound solution* nature has been disturbed because of the necessity of altering the load system at the nodes of the finite element mesh under the 'consistent load' lumping procedure.

As promised above, we now extend the concept of "best-fit" and variationally correct rate of convergence to the case of uniformly distributed load of intensity q on the cantilever with a little more effort. Now, when a finite element model is made, two levels of discretization error are introduced. Firstly, the uniformly distributed load is replaced by consistent loads, which are concentrated at element nodes. Thus, the first level of discretization error is due to the replacement of the quadratically varying bending moment in the actual beam with a linear bending moment within each beam element. Over the entire beam model, this variation is piecewise linear. The next level of error is due to the approximation implied in developing the stiffness matrix which we had considered above this effectively senses a "best-approximated" constant value of the bending moment within each element of the linear bending moment appearing to act after load discretization.

With these assumptions, it is a simple exercise using Castigliano's theorem and fictitious tip force and moment P and M respectively to demonstrate that the finite element model of such a problem using two-noded beam elements will yield a fortuitously correct tip deflection ($w = qL^4/8EI$) for all idealizations (i.e. even with one element!) and a tip rotation that converges at the rate $1/2N^2$ from above to the exact value ($\theta = qL^3/6EI$). Thus, as far as tip deflections are concerned, the two levels of discretization errors have nicely cancelled each other to give correct answers. This can deceive an unwary analyst into believing that an exact solution has been reached. Inspection of the tip rotation confirms that the solution is approximate and converging.

We see from the foregoing analysis that using linear Timoshenko beam elements for the tip loaded cantilever, the energy for this problem converges as $O(h^2)$ where $h=2l=L/N$ is the element size. We also see that this order of convergence carries over to the estimate of the tip deflections for this problem. Many text-books are confused over such relationships, especially those that proceed on the order of error analysis. These approaches arrive at conclusions such as strain error is proportional to element size, i.e. $O(h)$ and displacement error proportional to the square of the element size, i.e. $O(h^2)$ for this problem. We can see that for this problem (see Fig. 4.3) this estimate is meaningful if we consider the maximum error in strain to occur at element nodes (at centroids the errors are zero as these are optimal strain points). We also see that with element discretization, these errors in strain vanish as $O(h)$. We can also see that the strain energies are now converging at the rate of $O(h^2)$ and this emerges directly from the consideration that the discretized strains are 'best-fits' of the actual strain. This conclusion is not so readily arrived at in the order of error analysis methods, which often argue that the strains are accurate to $O(h)$, then strain energies are accurate to $O(h^2)$ because strain energy expressions contain squares of the strain. This conclusion is valid only for cases where the discretized strains are 'best-fit' approximations of the actual strains, as observed in the present example. If the 'best-fit' paradigm did not apply, the only valid conclusion that could be drawn is that the strains that have $O(h)$ error will produce errors in strain energy that are $O(2h)$.

4.4 The variationally correct rate of convergence

It is possible now to postulate that if finite elements are developed in a clean and strictly "legal" manner, without violating the basic energy or variational principles, there is a certain rate at which solutions will converge reflecting the fidelity that the approximate solution maintains with the exact solution. We can call this the variationally correct rate of convergence. However, this prescription of strictly legal formulation is not always followed. It is not uncommon to encounter extra-variational devices that are brought in to enhance performance.

4.5 Residual bending flexibility correction

The residual bending flexibility (RBF) correction [4.1,4.2] is a device used to enhance the performance of 2-node beam and 4-node rectangular plate elements (these are linear elements) so that they achieve results equivalent to that obtained by elements of quadratic order. In these C^0 elements (Timoshenko theory for beam and Mindlin theory for plate) there is provision for transverse shear strain energy to be computed in addition to the bending energy (which is the only energy present in the C^1 Euler-Bernoulli beam and Kirchhoff plate formulations). The RBF correction is a deceptively simple device that enhances performance of the linear elements by modifying the shear energy term to compensate for the deficient bending energy term so that superior performance is obtained. To understand that this is strictly an extra-variational trick (or "crime"), it is necessary to understand that a variationally correct procedure will yield linear elements that have a predictable and well-defined rate of convergence. This has been carried out earlier in this chapter. It is possible to improve performance beyond this limit only by resorting to some kind of extra-variational manipulation. By a variationally correct procedure, we mean that the stiffness matrix is derived strictly using a B^TDB triple product, where B and D are the strain-displacement and stress-strain matrices respectively.

4.5.1 The mechanics of the RBF correction.

MacNeal [4.2] describes the mathematics of the RBF correction using an original cubic lateral displacement field (corresponding to a linearly varying bending moment field) and what is called a *aliased* linearly interpolated field (giving a constant bending moment variation for the discretized field). We might recall that we examined the *aliasing* paradigm in Section 2.5 earlier. We find that a quicker and more physically insightful picture can be obtained by using Equations (4.5) and (4.6) above. Consider a case where only one element of length $2l$ is used to model a cantilever beam of length L with a tip load P . Applying Castigliano's theorem, we can show very easily that the continuum and discretized solutions will differ by,

$$w/L = P \left(L^2 / 3EI + 1/kAG \right) \quad (4.7)$$

$$\bar{w}/L = P \left(L^2 / 4EI + 1/kAG \right) \quad (4.8)$$

Note the difference in bending flexibilities. This describes the inherent approximation involved in the discretization process if all variational norms are strictly followed. The RBF correction proposes to manipulate the shear flexibility in the discretized case so that it compensates for the deficiency in the bending flexibility for this problem. Thus if k^* is the compensated shear correction factor, from Eq (4.7) and (4.8), we have

$$L^2/3EI + 1/kAG = L^2/4EI + 1/k^*AG$$

$$1/k^*AG = 1/kAG + L^2/3EI \quad (4.9)$$

It is not accurate to say here that the correction term is derived from the bending flexibility. The bending flexibility of an element that can represent only a constant bending moment (e.g. the two-noded beam element used here) is $L^2/4EI$. The missing $L^2/12EI$ (or $L^2/3EI$) is now brought in as a compensation through the shear flexibility, i.e. k is changed to k^* . This "fudge factor" therefore enhances the performance of the element by giving it a rate of convergence that is not variationally permissible. Two wrongs make a right here; or do they? Table 4.1 shows how the convergence in the problem shown in Fig. 4.2 is changed by this procedure.

4.6 Concluding remarks.

We have shown that the best-fit paradigm is a useful starting point for deriving estimates about errors and convergence. Using a simple example and this simple concept, we could establish that there is a variationally correct rate of convergence for each element. This can be improved only by taking liberties with the formulation, i.e. by introducing extra-variational steps. The RBF is one such extra-variational device.

It is also possible to argue that errors, whether in displacements, stresses or energies, due to finite element discretization must converge rapidly, at least in a $O(h^2)$ manner or better. If a large structure (domain) of dimension L is sub-divided into elements (sub-domains) of dimension l , one expects errors of the order of $(l/L)^2$. Thus, with ten elements in a one-dimensional structure, errors must not be more than a few percent. We shall discover however that a class of problems exist where errors are much larger - the discretizations fail in a dramatic fashion, and this cannot be resolved by the classical (pre-1977) understanding of the finite element method. A preliminary study of the issues involved will be taken up in the next chapter; the linear Timoshenko beam element serves to expose the factors clearly. Subsequent chapters will undertake a comprehensive review of the various manifestations of such errors. It is felt that this detailed treatment is justified, as an understanding of such errors has been one of the most challenging problems that the finite element procedure has faced in its history. Most of the early techniques to overcome these difficulties were *ad hoc*, more in the form of 'art' or 'black magic'. In the subsequent chapters, our task will be to identify the principles that establish the methodology underlying the finite element procedure using critical, rational scientific criteria.

4.7 References

- 4.1. R. H. MacNeal, A simple quadrilateral plate element, *Comput. Struct.*, 8, 175-183, 1978.
- 4.2. R. H. MacNeal, *Finite Elements: Their Design and Performance*, Marcel Dekker, NY, 1994.