INDUSTRIAL STATISTICS AND OPERATIONAL MANAGEMENT

Chapter 3: Transportation Problem

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CHAPTER 3

TRANSPORTATION PROBLEM

3.1 Introduction :

The "transportation problem" refers to a special class of linear programming problems dealing with the distribution of single commodity from various sources of supply to various points of demand in such a manner that the total transportation costs are minimized. It was first studied by F. L. Hitchcock in 1941, then separately by T. C. Koopmans in 1947, and finally placed in the framework of linear programming and solved by simplex method by G. B. Dantzig in 1951. Since then, improved methods of solutions have been developed and the range of application has been steadily widened. It is now accepted as one of the important analytical and planning tool in business and industry.

Refer to the following table :

	M_1	M_2	Supplies (in tons)
W_1	5	8	15
W_2	4	10	5
Demand (in tons)	12	8	

We consider the shipment of steel from two warehouses W_1 and W_2 to two markets M_1 and M_2. The cost of shipping from warehouse W_i to market M_j is given in the i^{th} - row and j^{th} - column of the table. For example, the cost of shipping from W_2 to M_1 is c_{21} i.e. Rs 4 per ton.

The supplies (a_i) at the warehouses are listed at the right of the table; thus the supply at W_1 is 15 tons. The demands (b_j) at the markets are listed at the bottom of the table; thus the demand at M_1 is 12 tons. Note that here the sum of the supplies equals the sum of the demands. i.e. $\Sigma a_i = \Sigma b_j$ Such problems are called balanced transportation problems. Let x_{ij} be the amount in tons to be shipped from warehouse W_i to market M_j. The problem is to ship the steel in least expansive (minimum cost) way and in doing so completely exhaust the supplies at the warehouses and exactly satisfy the demands at the markets.

3.2 Formulation of a general transportation problem :

Let us assume in general that there are m - sources $S_1, S_2, ..., S_m$ with capacities $a_1, a_2, ... , a_m$ and n - destinations (sinks) with requirements $b_1, b_2, ..., b_n$ respectively. The transportation cost from i^{th} - source to the j^{th} - sink is c_{ij} and the amount shipped is x_{ij}. If the total capacity of all sources is equal to the total requirement of all destinations, what must be the values of x_{ij} with i = 1, 2, ..., m and j = 1, 2, ..., n for the total transportation cost to be minimum ?

		(Sink)Destination					availability
Source		D_1	D_2			D_n	a_i
	S_1	c_{11}	c_{12}			c_{1n}	a_1
	S_2	c_{21}	c_{22}			c_{2n}	a_2
	:	:	:			:	:
	:	:	:			:	:
	S_m	c_{m1}	c_{m2}			c_{mn}	a_m
Demand (b_j)		b_1	b_2			b_n	$\sum a_i = \sum b_j$

Upon examining the above statement of the problem, we realize that it has an objective function which is

$$f(x) = c_{11}\, x_{11} + ... + c_{21}\, x_{21} + ... + c_{2n}\, x_{2n} + ... + c_{m1}\, x_{m1} + ... + c_{mn}\, x_{mn}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Secondly, in view of the condition that the total capacity is equal to the total requirement, i.e. $\sum a_i = \sum b_j$, the individual capacity of each source must be fully utilized and the individual requirement of each destination must likewise be fully satisfied. Hence we have m capacity constraints and n requirements constraints. The capacity constraints impose on the solution the condition that the total shipments of all destinations from any source must be equal to the capacity of that source. Thus,

$$x_{i1} + x_{i2} + ... + x_{in} = a_i, \qquad i = 1, 2, ..., m.$$

On the other hand, the requirement constraints require that the demand of every destination be fully satisfied by the total shipments from all sources. Thus,

$$x_{1j} + x_{2j} + ... + x_{mj} = b_j, \qquad j = 1, 2, ..., n.$$

Thirdly, there are the usual non-negativity constraints, i.e. $x_{ij} \geq 0$ for all i and j. They are based on the practical aspect that either we shall send some positive quantity or no quantity from any source to any sink.

To sum up, we have the following mathematical formulation of the transportation problem :

Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ (3.1)

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, 2, ...m \qquad (3.2)$$

$$\sum_{i=1}^{m} x_{ij} = bj \quad j = 1, 2, ...n \qquad (3.3)$$

and $x_{ij} \geq 0$ for all i and j (3.4)

The above formulation looks like an LPP. This special LPP will be called a **Transportation Problem** (TP).

3.2.1 Matrix form of a TP We can write (3.1) - (3.4) in matrix form as

$$\text{Minimize } z = c\,x\,,\ c,\ x^T \in R^{mn}$$

subject to $Ax = b$, $x \geq 0$, $b^T \in R^{m+n}$

where $x = [x_{11} ... x_{1n}\ x_{21} ... x_{2n}\ x_{m1} ... x_{mn}]$

$b = [a_1\ a_2 ... a_m\ b_1\ b_2 ... b_n]$

and A is a $(m + n) \times (mn)$ real matrix containing the coefficients of constraints and c is the cost vector. The elements of A are either 0 or 1. Thus, a LPP can be reduced to a TP if

1. the a_{ij} `s are restricted to the values 0 and 1.
2. The units among the constraints are homogeneous.

3.3 Types of transportation problem :

The TP can be classified into balanced TP or unbalanced TP.

a) Balanced transportation problem : If the sum of the supplies of all the sources is equal to the sum of the demands of all the destinations, then the problem is termed as a balanced TP. Here, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} bj$

b) Unbalanced transportation problem : If the sum of the supplies of all the sources is not equal to the sum of the demands of all the destinations, then the problem is termed as a unbalanced TP. Here, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} bj$

An unbalanced TP can be modified to a balanced one by introducing a dummy sink (destination) if $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} bj$ and a dummy source if $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} bj$.

The inflow from the source to a dummy sink represents the surplus at the source. Similarly, the flow from the dummy source to a sink represents the unfilled demand at the sink. The costs of transporting a unit from a dummy source to a dummy sink are assumed to be zero. The resulting problem is now balanced and can be solved.

For example consider the following problem :

		Sinks			
Sources		D1	D2	D3	Supply (ai)
	O1	30	50	15	300
	O2	35	70	20	200
	O3	20	45	60	500
	Demand(bj)	300	200	400	

Here $\sum_{i=1}^{3} a_i = 1000$ and $\sum_{j=1}^{3} bj = 900$. Thus it is an unbalanced TP. Here $\sum_{i=1}^{3} a_i > \sum_{j=1}^{3} bj$. Thus there is excess supply. So we have to include a dummy sink to absorb this excess supply of 100 units ($\sum_{i=1}^{3} a_i - \sum_{j=1}^{3} bj = 100$). The cost coefficients in the dummy destination are assumed to be zeroes. Modifying the given table yields a balanced TP as follows :

		Sinks				
Sources		D1	D2	D3	D4	Supply (ai)
	O1	30	50	15	0	300
	O2	35	70	20	0	200
	O3	20	45	60	0	500
	Demand(bj)	300	200	400	100	∑ai =∑bj = 1000

Similarly, an unbalanced TP can be converted to a balanced TP by adding a dummy source when the demand at the sink is greater than the supply at the source.

3.4 Some Theorems :

Theorem 3.1 : (Existence of a feasible solution)

The necessary and sufficient condition for the existence of a feasible solution to the TP is ∑ai =∑bj , that is, the total capacity (supply) must equal total requirement (demand).

Proof : (Necessary condition)

Let there exist a feasible solution to the TP given in (3.1) to (3.4). Then $\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

and $\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$

Therefore, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

(Sufficient condition)

Let us assume that . $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

We think of a working rule to distribute the supply at the i[th] - source in strict proportion to the requirements of all destinations; i.e. let $x_{ij} = \lambda_i b_j$, where λ_i is the proportionality factor for the i[th] - source.

Since this supply must be completely distributed,

$$\sum_{i=1}^{m} x_{ij} = \lambda_i \sum_{j=1}^{n} b_j = a_i$$

or $\lambda_i = \dfrac{a_i}{\sum_{j=1}^{n} b_j} = \dfrac{a_i}{\sum_{i=1}^{m} a_i}$ (given)

Thus, $x_{ij} = \lambda_i b_j = \dfrac{a_i}{\sum_{j=1}^{n} b_j} b_j$ (8.4.1.1)

Now, $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \dfrac{a_i}{\sum_{j=1}^{n} b_j} b_j = \dfrac{a_i}{\sum_{j=1}^{n} b_j} \sum_{j=1}^{n} b_j = a_i$

and $\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \dfrac{a_i}{\sum_{j=1}^{n} b_j} b_j = \dfrac{b_j}{\sum_{i=1}^{m} a_i} \sum_{i=1}^{m} a_i = b_j$

which shows that all the constraints of (3.1) - (3.4) are satisfied. Furthermore, since all ai and bj are positive, xij determined from (3.4.1.1) must be all positive. Therefore, (3.4.1.1) yields a feasible solution.

Theorem 3.4.2 :

The dimensions of the basis of a TP are (m + n - 1) × (m + n - 1). This means that a TP has only (m + n - 1) - independent structural constraints and its basic feasible solution has only (m + n - 1) - positive components.

Proof : It can be seen from the mathematical model of a TP that there are m - rows (capacity constraint equations) and n - columns (requirement constraint equations). Thus there are total (m + n) - constraint equations. This is due to the condition that . $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

For instance, suppose that we add together the m capacity constraints in (3.2),

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \qquad (3.4.2.1)$$

Now let us add the first (n - 1) - requirement constraints in (3.3),

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \qquad (3.4.2.2)$$

Now subtract (3.4.2.2) from (3.4.2.1). Then $\sum_{i=1}^{m} x_{in} = b_n$ which is the last requirement constraint.

Hence, one of the (m + n) - constraints can always be derived from the remaining (m + n - 1) - constraints and the problem in effect has only (m + n -1) - independent constraints.

Remarks :

1. This implies that from mn - variables of type x_{ij}, (m + n - 1) are basic variables and remaining mn - (m + n - 1) are non - basic variables. If at least one of these (m + n - 1) - variables take up zero value then the solution is called degenerate solution.
2. The allocated cell in the transportation table will be called occupied cells (or basic cells) and the empty cells will be called non - occupied cells.

3.4.1 Triangular basis :

We know that the number of basic variables is equal to the number of constraints in linear programming. In the same way when capacity and requirement constraints are expressed in terms of basic variables and all non - basic variables are given zero value, the matrix of coefficients of variables in the system of equations is triangular. This means that there is a row or a column in which there is only one basic variable. There is another row or column in which there are two basic variables, and so on.

Theorem 3.4.4 : The transportation problem has a triangular basis.

Proof : Initially, we observe that there is no equation in which there is not at least one basic variable. i.e. every equation has a basic variable otherwise, the equation cannot be satisfied for $a_i \neq 0$ or $b_j \neq 0$.

Suppose, every row and column equation has at least two basic variables. Since there are m rows and n columns, the total number of basic variables in row equations and column equations will be at least 2m and 2n respectively. Suppose, if the total number of basic variables is B, then obviously $B \geq 2m$, $B \geq 2n$. There can be three cases now,

Case (i) : $m > n$

$$m + m > m + n \Rightarrow 2m > m + n \Rightarrow B \geq 2m > m + n$$

Case (ii) : $m < n$

$$m + n < n + n \Rightarrow m + n < 2n \Rightarrow B \geq 2n > m + n$$

Case (iii) : $m = n$

$$m + m = m + n \Rightarrow 2m = m + n \Rightarrow B \geq 2m = m + n$$

Thus, in all cases $B \geq m + n$. But we know by theorem 8.4.2 that the number of basic variables in TP is $(m + n - 1)$. Thus, we have a contradiction. So our supposition that every equation has at least two basic variables is wrong. Therefore, there is at least one equation, either row or column, having only one basic variable.

Let the r^{th} - equation have only one basic variable and let x_{rs} be the only basic variable in the row r and column s, then $x_{rs} = a_r$. Eliminate r^{th} - row from the system of equation and substitute $x_{rs} = a_r$ in s^{th} - column equation and replace b_s by $b_s' = b_s - a_r$ (as explained in theorem 8.4.3).

After eliminating the r^{th} - row, the system has $(m - 1)$ - row equations and n column equations of which $(m + n - 2)$ are linearly independent. This implies that the number of basic variables is $(m + n - 2)$. Repeat the argument given earlier and conclude that in the reduced system of equations, there is an equation which has only one basic variable. But if this equation happens to be the s^{th} - column equation in the original system, then it will have two basic variables. This suggests that in our original system of equations, there is an equation which has at least two basic variables.

Continue the process to prove the theorem.

3.5 Solving the Transportation Problem (Finding initial basic feasible solution) :

The basic approach in solving the transportation problem is infact the same as that employed by the simplex method. First, an initial basic feasible solution is obtained. Then a new and better feasible solution is determined by means of the substitution of a basis variable by a non - basis variable, which tends to improve the value of the objective function.

In the TP, a basis variable identifies a route (from specific source to specific sink) that is in use while a non - basis variable identifies a route that is not in use. The problem is a composite of the following three parts:

1. How can non - basis routes be evaluated for their effects on the transportation costs ?
2. How can a favorable non - basis route be inserted into the basis to obtain a new basic feasible solution ?
3. How can an optimal solution be recognized so that the iteration may be ended ?

We now start by looking at methods to find the initial basic feasible solution.

3.5.1 Why using simplex method to solve a TP is unwise ?

Consider the following TP :

Example 3.1 An ABC car company has warehouses in Calcutta (W_1), Patna (W_2) and Bhavnagar (W_3) and markets in New Delhi (M_1), Dhanbad (M_2) and Calicut (M_3). At a particular time, the company has 61 cars in Calcutta, 49 in Patna and 90 in Bhavnagar. The company plans to transport 52 cars to New Delhi, 68 to Dhanbad and 80 to Calicut. The transportation costs per unit (in rupees) as well as the above data are given in the following table.

		Markets			
		M1	M2	M3	Supply
Warehouses	W1	26	23	10	61
	W2	14	13	21	49
	W3	16	17	29	90
	Demand	52	68	80	200

Formulating the above problem as a LPP, we obtain

Minimize $26x_{11} + 23x_{12} + 10x_{13} + 14x_{21} + 13x_{22} + 21x_{23} + 16x_{31} + 17x_{32} + 29x_{33}$

subject to

$$x_{11} + x_{12} + x_{13} = 61$$

$$x_{21} + x_{22} + x_{23} = 49$$

$$x_{31} + x_{32} + x_{33} = 90$$

$$x_{11} + x_{21} + x_{31} = 52$$

$$x_{12} + x_{22} + x_{32} = 68$$

$$x_{13} + x_{23} + x_{33} = 90$$

x_{11} , x_{12} , x_{13} , x_{21} , x_{22} , x_{23} , x_{31} ,x_{32} , $x_{33} \geq 0$.

The problem has 6 constraints and 9 variables. It will be unwise if not possible to solve such a problem using simplex method. This is why a special computational procedure is necessary to solve the transportation problem.

Now, we study the following three methods to find the initial basic feasible solution to a TP.

1. The North - West Corner Rule (NWCR)
2. Least Cost cell method (LCM)
3. Vogel's Approximation Method (VAM)

3.5.2 The North - West Corner Method (NWCM):`

Algorithm

Step 1 : Locate the cell (p. q) in the north - west (upper left) corner of the matrix of the data completely ignoring the transportation cost.

Step 2 : Transport the minimum of the supply and demand values with respect to that cell and subtract this minimum from the supply and demand values. Thus, if x_{pq} is minimum of a_p and b_q , then give x_{pq} to the cell (p, q). Replace a_p by $a_p - x_{pq}$ and b_q by $b_q - x_{pq}$..

Step 3 : Check whether exactly one of the row / column corresponding to the north - west corner cell has now zero supply / demand respectively. If yes, go to step 4, otherwise, go to step 5.

Step 4 : Delete that row / column with respect to the north - west corner cell which has the zero supply / demand and go to step 6.

Step 5 : Delete both the row and column with respect to the current north - west corner cell.

Step 6 : Check whether exactly one row or column is left out. If yes, go to step 7, otherwise, go to step 1.

Step 7 : Match the supply / demand of that row / column with the remaining demands / supplies of the undeleted columns / rows.

Step 8 : The North - West Corner rule is over. Go to the next phase of solving the TP.

It is clear that as soon as a value of x_{ij} is determined, a row or a column is eliminated from further considerations. The last value of xij eliminates both a row and a column. Hence, a feasible solution computed by North - West Corner rule can have at most (m + n -1) - positive x_{ij} if the TP has m - origins and n - destinations. Thus, the solution is a basic feasible solution.

Example 3.2 Find initial basic feasible solution for TP given in Example 3.1 by NWCR. Consider the TP given in section 3.5.1.

	M2	M3	
W1	9 23	10	0
W2	13	21	49
W3	17	29	90
	59	80	

		Markets			
		M1	M2	M3	Supply
Warehouses	O1	26	23	10	61
	O2	14	13	21	49
	O3	16	17	29	90
	Demand	52	68	80	200

As the problem is balanced, we move on to find the initial basic feasible solution by NWCR.

Step 1 : The north - west corner cell is (1, 1)with a cost entry $c_{11} = 26$.

Step 2 : Out of the supply value 61 and the demand value 52, minimum 52 is allocated to cell (1, 1) and that 52 is subtracted from the supply and demand values respectively.

		Markets			
		M1	M2	M3	Supply
Warehouses	W1	52 26	23	10	9
	W2	14	13	21	49
	W3	16	17	29	90
	Demand	0	68	80	

Step 3 : The first column, now has demand zero. So we move to step 4.

Step 4 : Deleting that column, we go to step 6. Now the new data matrix is

	M2	M3	
W1	23	10	9
W2	13	21	49
W3	17	29	90
	68	80	

Step 1

gives us the new north - west cell as the cell corresponding to W_1 and M_2 i.e. with entry 23. We again allocate 9 (minimum of 9 and 68) to that cell and subtract 9 from the demand and supply values yielding after step 2 the following table

Step 2 : First row has supply zero. So we move to step 4.

Step 4 : After deleting that row, we get

	M2	M3	
W2	13	21	49
W3	17	29	90
	59	80	

As still, two rows and columns are left, we go to step 1 again and get the north - west corner cell as the cell with entry 13 and repeat the steps as above.

	M2	M3	
W2	$\boxed{49}$ 13	21	0
W3	17	29	90
	10	80	

Now again the supply for second row is 0. Also according to step 6, we have only one row left at last.

	M2	M3	
W3	17	29	90
	10	80	

So we give 10 to the cell with entry 17 and 80 to the cell with entry 29 and stop.

	M2	M3	
W3	$\boxed{10}$ 17	$\boxed{80}$ 29	0
	0	0	

Thus, the full data table is now allocated as :

		Markets			
		M1	M2	M3	Supply
Warehouses	O1	$\boxed{52}$ 26	$\boxed{9}$ 23	10	0
	O2	14	$\boxed{49}$ 13	21	0
	O3	16	$\boxed{10}$ 17	$\boxed{80}$ 29	0
	Demand	0	0	0	

We read off the initial basic feasible solution from the allocated entries :

$x_{11} = 52$, $x_{12} = 9$, $x_{22} = 49$, $x_{32} = 10$, $x_{33} = 80$ and the other $x_{ij} = 0$.

The cost associated with this basic feasible solution is computed as follows :

$$C = 26(52) + 23(9) + 13(49) + 17(10) + 29(80) = \text{Rs. } 4686.$$

The set B of (m + n - 1) cells corresponding to the possible non - negative shipments (x_{ij}'s) of any basic feasible solution will be called the basis to the basic feasible solution. Thus, for this problem the basis is

$$B = \{ (1, 1), (1, 2), (2, 2), (3, 2), (3, 3) \}.$$

Note : Instead of writing and drawing tables stepwise as shown above, the whole method can be performed on a single data matrix as shown in the following example.

Example 3.3 : Obtain the initial basic feasible solution for TP by North - West Corner rule.

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1	1	9	13	36	51	50
F_2	24	12	16	20	1	100
F_3	14	35	1	23	26	150
Requirements	100	70	50	40	40	300

Solution : We first check whether the problem is balanced or not ? It is, so we proceed with the North - West Corner cell (1, 1).

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1	50 1	9	13	36	51	50 / 0
F_2	50 24	50 12	16	20	1	100 / 50 /0
F_3	14	20 35	50 1	40 23	40 26	150 /130 / 80 / 40 /0
Requirements	100 /50 / 0	70 / 20 / 0	50 / 0	40 / 0	40 / 0	

Since 50 < 100, we ship 50 by cell (1, 1); we slash the row 1 with a zero (/ 0) since b_1 has been reduced to zero. In addition, a_1 is reduced to 50. The remaining matrix (after neglecting first row), has the cell (2, 1) (corresponding to F_2 - R_1) as its north - West corner cell. Again $a_1 = 50 < b_2 = 100$. So we ship 50 by the cell (2, 1) and now $b_2 = 50$ and $a_1 = 0$. So we neglect column 1 and proceed. The cell (2, 2) is now the north - west corner cell for the new table and $b_2 = 50$, $a_2 = 70$ ($b_2 < a_2$). So we ship 50 by cell (2, 2) and now $a_2 = 20$, $b_2 = 0$. So we neglect row 2 and proceed.

Finally the initial basic feasible solution is :

$$x_{11} = 50, x_{21} = 50, x_{22} = 50, x_{32} = 20, x_{33} = 50, x_{34} = 40, x_{35} = 40.$$

The corresponding transportation cost is given by

C = 50 (1) + 50 (24) + 50 (12) + 20 (35) + 50 (1) + 40 (23) + 40 (26)

= Rs. 4560.

3.5.3 The Least - Cost (Matrix Minima) Method (LCM) :

Algorithm :

Step 1 : Find the minimum of the (undeleted) values in the cost matrix (i.e. find the matrix minima).

Step 2 : Find the minimum of the supply and demand values (X) with respect to the cell corresponding to the matrix minimum.

Step 3 : Allocate X - units to that cell. Also subtract X - units from the supply and demand values of that cell.

Step 4 : Check whether exactly one of row / column corresponding to that cell has zero supply / demand respectively. If yes, go to step 5, otherwise, go to step 6.

Step 5 : Delete that row / column with respect to the cell with the matrix minimum which has zero supply / zero demand and go to step 7.

Step 6 : Delete both the row and column with respect to the cell with the matrix minimum.

Step 7 : Check whether exactly one row or column is left out ? If yes, go to step 8 otherwise, go to step 1.

Step 8 : Match the supply / demand of that row / column with the remaining demand supplies of the undeleted rows / columns.

Step 9 : The least cost method is over. Go for the next phase of solving the TP.

Remark : In case the minimum cost cell is not unique, then select the cell where maximum allocation can be made.

The minimum transportation cost obtained by the matrix minimum method is much lower than the corresponding cost of the solution derived by using NWCR. This is to be expected as matrix minimum method takes into account the unit transportation cost while choosing the values of basic variables.

Example 3.4 : Find the initial basic feasible

		Markets			
		M1	M2	M3	Supply
Warehouses	O1	26	23	10	61
	O2	14	13	21	49
	O3	16	17	29	90
	Demand	52	68	80	200

solution for TP given in example 3.1 by least cost method.

Step 1 : We identify the least cost cell as cell (1, 3) with cost 10.

Step 2 : $a_1 = 61$, $b_3 = 80$, mini {61, 80} = 61.

Step 3 : We allocate 61 in cell (1, 3) and new $a_1 = 0$, $b_3 = 80 - 61 = 19$. Thus, we get $x_{13} = 61$.

Step 4 : As $a_1 = 0$, moving to step 5, we delete that row and obtain

	M1	M2	M3	
W2	14	13	21	49
W3	16	17	29	90
	52	68	19	

As still, two rows are left, we again identify the least cost cell in this new matrix. It is the cell with the entry 13.

Proceed as above,

	M1	M2	M3	
W2	14	$\boxed{49}$ 13	21	0
W3	16	17	29	90
	52	19	19	

Again after deleting the second row of W_2, we have only one row left. Hence,

	M1	M2	M3	
W3	$\boxed{52}$ 16	$\boxed{19}$ 17	$\boxed{19}$ 29	0
	0	0	0	

So, the final table looks like

		Markets			
		M1	M2	M3	Supply
Warehouses	W1	26	9	$\boxed{61}$ 10	61 / 0
	W2	14	$\boxed{49}$ 13	21	49 / 0
	W3	$\boxed{52}$ 16	$\boxed{19}$ 17	$\boxed{80}$ 29	90 / 0
	Demand	52 / 0	68 / 19 /0	80 /19 /0	

So the initial basic feasible solution is :

$x_{13} = 61, x_{22} = 49, x_{31} = 52, x_{32} = 19, x_{33} = 19.$

and the transportation cost is

$C = 61 (10) + 49 (13) + 52 (16) + 19 (17) + 19 (29) =$ Rs. 2953.

Example 3.5 : Find the initial basic feasible solution to the TP given in example 8.3 by LCM.

Solution :

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1	1	9	13	36	51	50
F_2	24	12	16	20	1	100
F_3	14	35	1	23	26	150
Requirements	100	70	50	40	40	300

We select the cell (1, 1) with entry 1. Note that here there is a tie between cells (1, 1), (2, 5) and (3, 3) all having costs 1. But in cells (1, 1) and (3, 3), we can allocate 50 units and in cell (2, 5), 40 units. So we select cell (1, 1) to start. (You can also start with (3, 3).) Then we allocate in cell (3, 3) and at last to cell (2,5).

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1	50 1	9	13	36	51	50 / 0
F_2	24	12	16	20	40 1	100 / 60
F_3	14	35	50 1	23	26	150 /100
Requirements	100 /50	70	50 / 0	40	40 / 0	

As new $a_1 = 0$ and new $b_3 = b_5 = 0$, we neglect them and get reduced TP matrix as

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1						
F_2	24	12		20		60
F_3	14	35		23		100
Requirements	50	70		40		

Proceeding, we start with cell (2,3) and get

	R_1	R_2	R_3	R_4	R_5	Capacity
F_2	24	60 12		20		60 / 0
F_3	50 14	10 35		40 23		100 / 50 /10 /0
Requirements	50 / 0	70 / 10 /0		40 / 0		

This stops the method. The overall allocations can be viewed as

	R_1	R_2	R_3	R_4	R_5	Capacity
F_1	50 1	9	13	36	51	50 / 0
F_2	24	60 12	16	20	40 1	100 / 60 / 0
F_3	50 14	10 35	50 1	40 23	26	150 / 100 / 50 /10 /0
Requirements	100 /50 / 0	70 / 10 /0	50 / 0	40 / 0	40 / 0	

Thus, the initial basic feasible solution is :

$x_{11} = 50$, $x_{22} = 60$, $x_{25} = 40$, $x_{31} = 50$, $x_{32} = 10$, $x_{33} = 50$, $x_{34} = 40$.

and the initial minimum transportation cost is

$C = 50 (1) + 60 (12) + 40 (1) + 50 (14) + 10 (35) + 50 (1) + 40 (23) = 2830.$

(Note that this cost is less compared to that in example 3.3)

3.5.4 Vogel's approximation method (Penalty method) VAM :

The core of Vogel's method is the idea that a penalty will be incurred if the best (lowest cost) route is not used for a source or a destination. Thus, for each source and destination, we compute a 'penalty' rating which is the difference in cost of the two cheapest routes for that source or destination. This penalty rating is recorded for all sources and destinations. Now, we choose that source or destination with the highest rating to attend first on the theory that it is more important to avoid the high penalty associated with a wrong assignment there. In this method, allocations are made so that the penalty cost is minimized. The advantage of this method is that it gives an initial solution which is nearer to an optimal solution than those obtained with NWCR and LCM.

Algorithm :

Step 1 : From the transportation table, we determine the penalty for each row and column. The penalties are calculated for each row (column) by subtracting the lowest cost element in that row (column) from the next

		Markets			
		M1	M2	M3	Suppl y
Warehouses	O1	26	23	10	61
	O2	14	13	21	49
	O3	16	17	29	90
	Demand	52	68	80	200

lowest cost element in the same row (column). Write down the penalties below the rows (aside the columns) of the table.

Step 2 : Select the row (column) with the highest penalty rating and allocate as much as possible from the supply and requirement values to the cell having the minimum cost. If there is a tie in the values of penalties, then select that cell where least cost cell occurs. If there is a tie in the least cost entries of a selected row / column, select that entry for which maximum allocations can be made.

Step 3 : Adjust the supply and demand conditions for that cell. Eliminate those rows (columns) for which the supply and demand requirements are met.

Step 4 : Repeat the above steps until the entire available capacity at various sources and requirements at various destinations are met.

Thus, we obtain an initial basic feasible solution.

Example 3.6 : Calculate the initial basic feasible solution for the TP in example 3.1 by VAM.

Solution :

Step 1 : First, we calculate the penalties for all sources and destinations. They are written below the table for the row differences and aside the table for the column differences.

		Markets				
		M1	M2	M3	a_i	penalty
	W 2	14	13	21	4 9	1
	W 3	16	17	29	9 0	1
Demand		52	68	19		
penalty		2	4	**8**		

Step 2 : We now select row 1 because it has the highest penalty rating 13. Now we look at the supply and demand values, corresponding to the cell with the least value in row 1. Here, it is the cell (1, 3) with entry 10 and a_1 = 61, b_3 = 80. So we allocate 61 to (1, 3).

Step 3 : We also adjust the new supply and demand a_1 and b_3 as shown below :

		Markets				Row penalty
		M1	M2	M3	Supply	1
Warehouses	W1	26	23	10	61	**13**
	W2	14	13	21	49	1
	W3	16	17	29	90	1
	Demand	52	68	80		
column penalty	1	2	4	11		

As the capacity for source 1 is satisfied, we neglect that row from further consideration.

		Markets				Row penalty
		M1	M2	M3	Supply	1
Warehouses	W1	26	23	61 10	61 / 0	**13**
	W2	14	13	21	49	1
	W3	16	17	29	90	1
	Demand	52	68	80 / 19		
column penalty	1	2	4	11		

Here, we select the destination 3 i.e. at the 3rd column because it has the largest penalty and allocate the 19 units to the cell (2, 3) having the least cost 21.

	M1	M2	a_i
W2	14	30 13	0
W3	16	17	90
bj	52	68	

As the requirement constraint for the third destination is met, we neglect that from further consideration. Finding penalties for new reduced matrix,

		Markets				
		M1	M2	M3	a_i	penalty
	W2	14	13	19	21	30	1
	W3	16	17	29	90	1
Demand		52	68	0		
penalty		2	4	8		

Repeating the above procedure, we obtain

		M1	M2	a_i	penalty
	W2	14	13	30	1
	W3	16	17	90	1
Demand		52	68		
penalty		2	**4**		

Finally, we obtain

		M1	M2	a_i
	W3	52	16	38	17	0
Demand		0	0	

The whole procedure can be done in a single table in the following way :

		Markets				Row Penalty		
		M1	M2	M3	Supply	1	2	3
Ware houses	W1	26	23	61 10	61 / 0	**13**	--	--
	W2	14	30 13	19 21	49 /30 / 0	1	1	1
	W3	52 16	38 17	29	90 / 38 /0	1	1	1
Demand		52	68 / 38	80 /19 / 0				
Column penalty	1	2	4	11				
	2	2	4	**8**				
	3	2	**4**	--				

Thus, the initial basic feasible solution is :

$x_{13} = 61$, $x_{22} = 30$, $x_{23} = 19$, $x_{31} = 52$, $x_{32} = 38$

and the total transportation cost is computed as follows :

$$C = 61(10) + 30(13) + 19(21) + 52(16) + 38(17) = \text{Rs. } 2877.$$

Note : The same example when solved for initial basic feasible solution by the three different methods give

	Method used	Transportation cost
Ex 3.1	NWCR	4686
Ex 3.4	LCM	2953
Ex 3.6	VAM	2877

The reader can easily see that VAM gives better initial solution in terms of less transportation cost. However sometimes, by chance, NWCR or LCM may give better pfs.

Example 3.7 Calculate the initial basic feasible solution for the TP given in ex. 3.3 by VAM.

Solution :

		R_1	R_2	R_3	R_4	R_5	a_i	Row Penalties		
								1	2	3
F_1		$\boxed{50}$ 1	9	13	36	51	50 / 0	8	8	--
F_2		24	$\boxed{60}$ 12	16	20	$\boxed{40}$ 1	100 /60 /0	11	4	4
F_3		$\boxed{50}$ 14	$\boxed{10}$ 35	$\boxed{50}$ 1	$\boxed{40}$ 23	26	150 / 100 / 90 / 40 / 0	13	**13***	13
b_j		100 / 50 / 0	70 / 10 / 0	50 / 0	40 / 0	40 / 0				
Column penalty	1	13	3	12	3	**26**				
	2	**13***	3	12	3	--				
	3	10	**21**	15	3	--				

Firstly, checking the penalties for the first round, we find it is 26 for column R_5. So we allocate in its least cost cell with cost 1, the amount 40 of its requirements, leading to its new requirement being zero, thus neglecting the last column from future considerations.

Now finding the penalties again for the new reduced matrix (i.e. after neglecting the last column), we find there is a tie between penalties of F_3 and R_1 (both are 13, marked with *). So we select the least cost cell from both. In this case both have least cost cell as 1. So we arbitrarily select the cell (1, 1) i.e. corresponding to F_1 - R_1. We allocate 50 capacity values thereby exhausting it and the new requirement value is $b_1 = 50$. Now we neglect the first row also from future consideration.

We proceed similarly and obtain the initial basic feasible solution as

$x_{11} = 50$, $x_{22} = 60$, $x_{25} = 40$, $x_{31} = 50$, $x_{32} = 10$, $x_{33} = 50$, $x_{34} = 40$

and the minimum transportation cost is

$$C = 50(1) + 60(12) + 40(1) + 50(14) + 10(35) + 50(1) + 40(23) = \text{Rs. } 2830.$$

Note that in this case this value is less than that obtained by NWCR for the same table in example 3.3 but same as that of LCM in example 3.5.

We have seen the first stage of solving a transportation problem i.e. finding the initial basic feasible solution. We have also seen that VAM is more preferable to NWCR and LCM for finding the initial basic feasible solution. We now move ahead and see whether the initial basic feasible solution obtained is optimal or not. Thus, we have to check whether the set of allocations obtained in VAM are the beat possible to reduce the total transportation costs, or do any other set of such allocations exist ?

3.6 Loops in a transportation method :

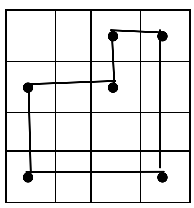
In a transportation table, an ordered set of four or more cells is said to form a loop if

1. any two adjacent cells in the ordered set lie either in the same row or in the same column, and
2. any three or more adjacent cells in the ordered set do not lie in the same row or the same column. The first cell of the set is considered to follow the last in the cell.

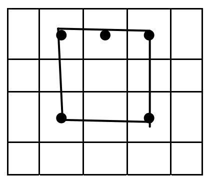
If we join the cells of a loop by horizontal and vertical segments, we get a closed path satisfying (1) and (2) above. Let us denote by (i, j), the (i, j)th cell of a transportation table, then, it can be seen from the tables below that the set

L_1 = {(2, 1), (4, 1), (4, 4), (1, 4), (1, 2), (2, 2)} forms a loop, while

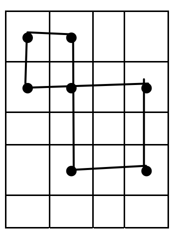
L_2 = {(3, 2), (3, 5), (2, 5), (2, 4), (2, 3), (1, 3), (1, 2)} does not form a loop.

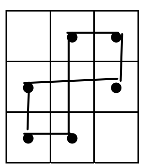


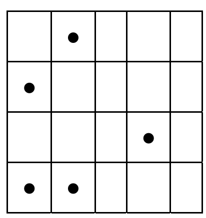
Loop L_1
Fig (a)



Non Loop L_2
Fig (b)







Non Loop Fig (c) Loop Fig (d) Fig (e)

The indication of independence of a set of individual positive allocations is that we cannot form a loop by joining positive allocations by horizontal and vertical lines only, for example see fig (a) and (d). In fig (a) and (d), a closed loop is formed by joining the positive allocation cells by horizontal and vertical lines. But in fig (e), we will not be able to form such a loop. Thus, allocations in fig (a) and (d) are independent in position while in fig (e) are dependent in position.

Note that every loop will contain even number of cells. Independent allocations have a property that it is impossible to increase or decrease any individual allocation without altering the positions of individual allocations or violating the row or column sum restrictions.

15 C11	C12	C13	15
C21	19 C22	C23	19
C31	C32	5 C33	5
15	19	5	

3.7 Optimality in a transportation problem :

After determining initial basic feasible solution the problem arises of recognizing an optimum solution i.e. whether or not the solution thus obtained minimizes the total transportation cost. For this we determine the value called ***NET EVALUATION CONTRIBUTIONS (OPPORTUNITY COST)*** , corresponding to each empty cell (cell where no allocations are made) of the transportation matrix. If a unit is allocated in an unoccupied cell and the adjustments are made in the solution to maintain the row and column sums, then the net change in the total cost resulting from the adjustments is called net evaluation contribution of that cell and is denoted by Δ_{ij}.

The unoccupied cell with the largest negative Δ_{ij} value is selected to include in the new set of allocations. It is incoming variable. The outgoing variable in the current solution is the occupied cell (basic variable) in the unique close path (loop) whose allocation will become zero first as more units are allocated to the cell with largest negative Δ_{ij} value. Such an exchange reduces total transportation cost. The process is continued until there is no negative Δ_{ij} value for all non allocated cells. That is, the current solution cannot be improved further. It is the optimal. We will discuss in detail as we move along.

Note : For example, consider the following table (for understanding the concept only)

Here allocations are made in cells (1, 1), (2, 2) and (3, 3). If we wish to find the net evaluations of the cell (1, 2) i.e. we increase the allocations of the cell (1, 2) from 0 to 5, then in order to satisfy the row - column sum restrictions, we have to decrease the allocation at cell (1, 1) by 5 and eventually the cell (2, 1) will also have 5 - allocations from the cell (2, 2).

10 C11	5 C12	C13	15
5 C21	14 C22	C23	19
C31	C32	5 C33	5
15	19	5	

Note that the adjustments are done only on the positive allocations and except these four allocations, other allocations remain unchanged and so their transportation cost will also remain unchanged. The <u>net change</u> in the total transportation cost due to the allocations in cell (2, 1) is

$$5c_{21} - 5\,c_{11} + 5\,c_{12} - 5\,c_{22}$$

This is the relative cost coefficient of the variable x_{21}. Similarly we can find such coefficients for all non-basic variables (empty cells).

3.7.1 Dual of a transportation problem :

Consider the general transportation problem defined in (8.1) - (8.4)

Minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1, 2, ..., n$$

and $\quad x_{ij} \geq 0$ for all i and j

Since all the constraints are equalities, we convert them into inequalities and reformulate the problem as

Minimize $\quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

subject to

$$\sum_{j=1}^{n} x_{ij} \geq a_i \quad , \ i = 1, 2, ..., m \tag{3.5}$$

$$\sum_{j=1}^{n} (-x_{ij}) \geq (-a_i) \quad i = 1, 2, ..., m \tag{3.6}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \quad , \ j = 1, 2, ..., n \tag{3.7}$$

$$\sum_{i=1}^{m} (-x_{ij}) \geq (-b_j) \quad , j = 1, 2, ..., n \tag{3.8}$$

and $\quad x_{ij} \geq 0$ for all i and j.

Let u_i^+ and u_i^- be the dual variables one for each capacity constraint i in (3.5) and (3.6) respectively. Let v_j^+ and v_j^- be the dual variables one for each requirement constraint j in (3.7) and (3.8) respectively.

Dual of the above problem is

Maximize $$z^* = \sum_{i=1}^{m} (u_i^+ - u_i^-) a_i + \sum_{j=1}^{n} (v_j^+ - v_j^-) b_j \tag{3.9}$$

subject to

$$(u_i^+ - u_i^-) + (v_j^+ - v_j^-) \leq c_{ij} \tag{3.10}$$

and u_i^+, u_i^-, v_j^+ and $v_j^- \geq 0$ for all i and j.

Let $u_i = u_i^+ - u_i^-$ and $v_j = v_j^+ - v_j^-$. Then u_i and v_j will be unrestricted in sign. Then problem in (3.9) - (3.10) can now be rewritten as

Maximize $$z^* = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j \tag{3.11}$$

subject to

$u_i + v_j \leq c_{ij}$ for all (i , j) (3.12)

and u_i , v_j unrestricted in sign for all i and j.

Remarks :

1. The variables u_i and v_j are the shadow prices for capacities and requirements respectively. They represent the implicit contribution (value) of an additional unit of capacity at source i and an additional unit transported to requirement j.

2. The variables x_{ij} forms an optimal solution to the given transportation problem provided

a) solution x_{ij} is feasible for all (i, j) with respect to (3.4).

b) solution u_i and v_j is feasible for all (i, j) with respect to (3.11) - (3.12).

c) $(c_{ij} - u_i - v_j)\, x_{ij} = 0$ for all i and j. This is the complementary slackness for a TP and indicate that

→ if $x_{ij} > 0$ and is feasible in (8.1) - (8.4) then $c_{ij} - u_i - v_j = 0$ or $c_{ij} = u_i + v_j$ for each occupied cell.

→ if $x_{ij} = 0$ and $c_{ij} > u_i + v_j$ then it is not desirable to have $x_{ij} > 0$ in the basis because it would cost more to transport on a route (i, j).

→ if $c_{ij} < u_i + v_j$ for some $x_{ij} = 0$ then x_{ij} can be brought into the basis.

3. Recall from LPP that for any standard LPP with basis B and associated cost vector c_B , the associated solution to its dual is given by $u = c_B B^{-1}$. Thus, if a_j is the j^{th} - column of the primal constraint matrix then an expression for evaluating the net evaluation is given by

$$c_j - z_j = c_j - c_B B^{-1} a_j = c_j - u\, a_j$$

In our present case of TP, the associated dual solution can be represented as $(u \;\; v) = (u_1, u_2, ..., u_m, v_1, v_2, ..., v_n)$ and hence net evaluations are given by

$$\Delta_{ij} = c_{ij} - z_{ij} = c_{ij} - (u \;\; v)\, a_{ij}$$

$$= c_{ij} - (u_1, u_2, ..., u_m, v_1, v_2, ..., v_n)\,(e_i + e_{m+j}) = c_{ij} - (u_i + v_j)$$

Here, it may be noted that $\Delta_{ij} = 0$ for occupied cells (basic variables). Except for the degeneracy case, there are (m + n -1) dual equations in (m + n) dual unknowns, for the (m + n -1) basic cells. We can assign arbitrary value for one of these unknown u_i and v_j and solve uniquely for the remaining (m + n -1) - variables. After an arbitrary assignment say $u_1 = 0$, rest of the values are obtained by simple addition and subtraction. (Here we will use a thumb rule that - that u_i or v_j should be taken as zero whose row / column has the maximum number of allocated cells.) Once all the u_i and v_j have been determined, the net evaluations for all the non - basic cells are obtained by using the relation $\Delta_{ij} = c_{ij} - (u_i + v_j)$.

3.7.2 Theorem : If we have a non - degenerate basic feasible solution i.e. a cell with exactly (m + n - 1) independent positive allocations and a set of arbitrary numbers u_i and v_j , i = 1, 2, ... , m and j = 1, 2, ... , n such that $c_{rs} = u_r + v_s$ for all occupied cell (r, s) then the net evaluations are given by $\Delta_{ij} = c_{ij} - (u_i + v_j)$.

Proof : The TP is to find $x_{ij} \geq 0$ so as to

Minimize $$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (3.7.2.1)$$

subject to

$$a_i - \sum_{j=1}^{n} x_{ij} = 0 \qquad , i = 1, 2, ..., m \qquad (3.7.2.2)$$

$$b_j - \sum_{i=1}^{m} x_{ij} = 0 \qquad , j = 1, 2, ..., n \qquad (3.7.2.3)$$

Now adding u_i times (3.7.2.2) and v_j times (3.7.2.3) to the objective function (3.7.2.1), it becomes

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} u_i (a_i - \sum_{j=1}^{n} x_{ij}) + \sum_{j=1}^{n} v_j (b_j - \sum_{i=1}^{m} x_{ij})$$

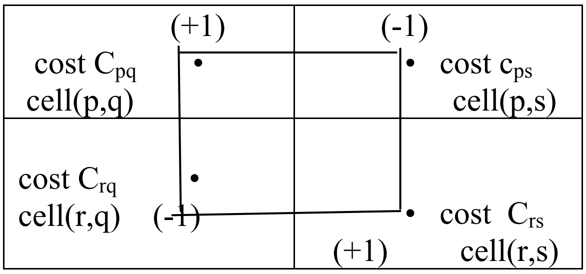
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \{c_{ij} - (u_i + v_j)\} x_{ij} + \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j \qquad (3.7.2.4)$$

But given that for all occupied (basic cells), $c_{rs} = u_r + v_s$, all terms of positive allocations vanish from (3.7.2.4) as their coefficients become zero.

Therefore,

$$z = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j \qquad (3.7.2.5)$$

Suppose we want to determine the net evaluation for the cell (p, q). As we allocate one unit to cell (p, q), positive allocations become (m + n) in number and hence dependent. So a loop can be formed as shown below :



Here all (p, s), (r, q), (r, s) are occupied cells hence $c_{ps} = u_p + v_s$, $c_{rq} = u_r + v_q$, $c_{rs} = u_r + v_s$. Now to maintain row and column sums, we have to decrease the individual allocations at (p, s) and (r, q) cells and increase at cell (r, s) by 1 unit. Therefore the value of the individual allocations in these occupied cells is changed but they contribute nothing in the objective function (8.7.2.4) as their coefficients are necessarily zero. Thus, the new values of the objective function corresponding to this new solution is given by

$$z^* = \{c_{pq} - (u_p + v_q)\} + \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$$

Hence, the net evaluation is given by

$$\Delta_{pq} = z^* - z = [c_{pq} - (u_p + v_q)] \qquad (3.7.2.6)$$

Thus, in general, $\Delta_{ij} = c_{ij} - (u_i + v_j)$.

3.8 Transportation algorithm (u - v method Modified distribution (MODI) method) :

Algorithm :

Step 1 : We find initial basic feasible solution by applying VAM on a balanced transportation problem. Thereby we find the initial transportation cost.

Step 2 : In order to check the optimality of the cost, we apply the optimality criteria : There must be (m + n - 1) number of allocated cells (m - number of rows, n - number of columns). If number of allocated cells is less than (m + n - 1), then it is a degenerate TP.

For allocated cells, we break-up the cost as $c_{ij} = u_i + v_j$. **(That u_i / v_j should be taken as zero, for row / column having maximum number of allocated cells).**

Step 3 : Now find the opportunity cost for non - allocated cells by $\Delta_{ij} = c_{ij} - (u_i + v_j)$.

Step 4 : Examine the sign for each Δ_{ij} .

i) If all $\Delta_{ij} \geq 0$, then the optimality criteria has been satisfied and the initial cost obtained is optimum. If some $\Delta_{ij} = 0$, then it is the case of alternative optimal solution.

ii) If some $\Delta_{ij} < 0$, then the cost obtained is not optimal. It has to be reduced by some technique (given in next step).

Step 5 : In the cell having the most negative value of Δ_{ij} , we will make allocations. Rescheduling of allocations is done by looping process.

Start the closed path with the selected unoccupied cell and mark (+) in this cell, trace a path along the rows (or columns) to an occupied cell with minimum allocations, mark (-) in this cell and continue down the column (or row) to an occupied cell and mark (+) and (-) signs alternatively. Close the path back to the selected unoccupied cell. All end points of the loop are allocated cells except the one which corresponds to the most negative Δ_{ij} entry. All (+) are ACCEPTOR cells and (-) are DONOR cells. (Hence a loop is always of even number of cells and it is unique.

Step 6 : Out of the donors, the one with the lowest allocation gives same number of allocated units to the most negative Δ_{ij} acceptor cell (+). Add that quantity to all other acceptor cells (+) and subtract it from the cells marked with (-) sign.

Step 7 : Now find the new minimum cost by allocating units to the remaining non-allocated cells.

Step 8 : Test the revised solution for optimality.

Note : Step 2 to step 8 above is known as Modified Distribution (MODI) method.

Example 3.8 Check the optimality of the solution obtained by VAM for the TP in example 3.6. If the solution is not optimal, modify it.

Solution : The initial basic feasible solution obtained in example 3.6 is as follows :

and the initial transportation cost is Rs. 2877.

Now, we apply the transportation algorithm (MODI Method) to check the optimality.

Step 1: We have already calculated.

Step 2 : Here the allocated cells (basic cells) are the cells (1, 3), (2, 2), (2, 3), (3, 1), (3, 2) where the allocations 61, 30, 19, 52, 38 respectively are made and they are 5 in number. Also m + n - 1 = 3 + 3 - 1 = 5. Therefore, number of allocated cells are 5 = (m + n -1).

		Markets				Row Penalty		
		M1	M2	M3	Supply	1	2	3
Ware houses	W1	26	23	61 10	61 / 0	**13**	--	--
	W2	14	30 13	19 21	49 /30 / 0	1	1	1
	W3	52 16	38 17	29	90 / 38 /0	1	1	1
Demand		52	68 / 38	80 /19 / 0				
Column penalty	1	2	4	11				
	2	2	4	**8**				
	3	2	**4**	--				

Now we break-up the cost for basic cells as $c_{ij} = u_i + v_j$ with thumb rule : That u_i / v_j should be taken as zero, for which maximum number of allocations are made in its row / column.

		10	$u_1 =$
	13	21	$u_2 = 0$
16	17		$u_3 =$
$v_1 =$	$v_2 =$	$v_3 =$	

1	2		$u_1 = -11$
12			$u_2 = 0$
		25	$u_3 = 4$
$v_1 = 12$	$v_2 = 13$	$v_3 = 21$	
($u_i + v_j$) addition for non allocated cells from the step 2			

Let us take $u_2 = 0$.

As $u_2 + v_2 = 13$, $v_2 = 13$

As $v_2 + u_3 = 17$, $u_3 = 4$

As $u_3 + v_1 = 16$, $v_1 = 12$

As $u_2 + v_3 = 21$, $v_3 = 21$

As $u_1 + v_3 = 10$, $u_1 = - 11$

Thus, we get

		10	$u_1 = -11$
	13	21	$u_2 = 0$
16	17		$u_3 = 4$
$v_1 = 12$	$v_2 = 13$	$v_3 = 21$	

Step 3 : We now find the opportunity cost for non-allocated cells (non - basic cells) by

$\Delta_{ij} = c_{ij} - (u_i + v_j)$ i.e. we have u_i / v_j values from previous step for all those cells, where allocations are not made, we will subtract $(u_i + v_j)$ from the initial cost given in the initial TP cost matrix.

26	23	
14		
		29
Cost values given in the original matrix for non allocated cells		

Thus Δ_{ij} values are $c_{ij} - (u_i + v_j)$ as shown :

25	21	
2		
		4

Step 4 Examining the sign of each Δ_{ij} . As each $\Delta_{ij} > 0$, the optimality criteria is satisfied and the cost obtained is optimum.

Thus, here the initial cost obtained by applying VAM is optimum.

Example 3.9 : Check the optimality of the solution obtained by VAM for TP in example 3.7. If the solution is not optimal, modify it.

Solution : The initial basic feasible solution obtained in example 3.7 is :

		R_1	R_2	R_3	R_4	R_5	a_i	Row Penalties 1	2	3
F_1		$\boxed{50}$ 1	9	13	36	51	50 / 0	8	8	--
F_2		24	$\boxed{60}$ 12	16	20	$\boxed{40}$ 1	100 /60 /0	11	4	4
F_3		$\boxed{50}$ 14	$\boxed{10}$ 35	$\boxed{50}$ 1	$\boxed{40}$ 23	26	150 / 100 / 90 / 40 / 0	13	**13***	13
b_j		100 / 50 / 0	70 / 10 / 0	50 / 0	40 / 0	40 / 0				
Column penalty	1	13	3	12	3	**26**				
	2	**13***	3	12	3	--				
	3	10	**21**	15	3	--				

and the initial transportation cost is Rs. 2830.

Step 1 : Calculated.

Step 2 : Number of allocated cells are 7 = (m + n -1) = 3 + 5 - 1. Thus number of allocated cells is equal to (m + n -1). Break-up the cost for basic cells as $c_{ij} = u_i + v_j$

1					$u_1 = -13$
	12			1	$u_2 = -23$
14	35	1	23		$u_3 = 0$
$v_1 = 14$	$v_2 = 35$	$v_3 = 1$	$v_4 = 23$	$v_5 = 24$	

As u_3 corresponds to a row which has the maximum number of cells are allocated (4 cells are allocated) , we take $u_3 = 0$ and then $v_1 = 14$, $v_2 = 35$, $v_3 = 1$, $v_4 = 23$, $v_5 = 24$, $u_1 = -13$, $u_2 = -23$.

Step 3 : We now find the opportunity cost for non-allocated cells by $\Delta_{ij} = c_{ij} - (u_i + v_j)$

	9	13	36	51
24		16	20	
				26
Cost values given in the original matrix for non allocated cells				

	22	-12	10	11
-9		-22	0	
				24
$(u_i + v_j)$ addition for non allocated cells from the step 2				

Thus, $\Delta_{ij} = c_{ij} - (u_i + v_j)$ are as shown as below :

	-13	25	26	40
33		38	20	
				2

Hence the value of Δ_{ij} in the cell (1, 2) is - 13 which is negative. This indicates that the initial basic feasible solution and hence the transportation cost obtained is not optimal.

Step 5 : We have to make allocations in the cell where Δ_{ij} is most negative i.e. in the cell (1, 2). We have to trace a closed path from this cell (non-basic) to the other cell which are basic cells. (In the above table, it is clear that those cells which are empty in Δ_{ij} - table are basic cells because we find Δ_{ij} only for non-basic cells.) Thus, a closed path can be easily identified. While making such a closed path we will also alternatively assign '+' and '-' sign indicating that all '+' are acceptor cells and '-' are donor cells. The traced path from above table is identified as

(-) •	• (+) -13	25	26	40
33		38	20	
(+) •	• (-)			
(-) •	• (+) -13	25	26	40
33		38	20	
(+) •	• (-)			

We now make a new table indicating the basic cells and the cell with negative Δ_{ij} entry

50 (-) D 1 •	• (+)A			
	60 12			40 1
50 14 (+)A •	10 (-)D • 35	50 1	40 23	

Now out of both the donors; cell (1, 1) and (3, 2), the one having lowest allocation gives 10 units to the cell (1, 2). Similarly to balance this, the donor cell (1, 1) gives 10 units to the acceptor cell (3, 1). Thus the new allocation table is

40 1	10 9			
	60 12			40 1
60 14		50 1	40 23	

and the new cost associated with it is

40 (1) + 10 (9) + 60 (12) + 40 (1) + 60 (14) + 50 (1) + 40 (23) = Rs. 2700.

Therefore new feasible solution is

$x_{11} = 40$, $x_{12} = 10$, $x_{22} = 60$, $x_{25} = 40$, $x_{31} = 60$, $x_{33} = 50$, $x_{34} = 40$.

Now again we need to check its optimality.

number of allocated cells = 7 = m + n - 1.

Break-up the cost for basic cells as $c_{ij} = u_i + v_j$

1	9				$u_1 = -13$
	12			1	$u_2 = -10$
14		1	23		$u_3 = 0$
$v_1 = 14$	$v_2 = 22$	$v_3 = 1$	$v_4 = 23$	$v_5 = 11$	

Calculate opportunity cost for non-allocated cells

		13	36	51
24		16	20	
	35			26
Cost values given in the original matrix for non allocated cells				

		-12	10	-2
4		-9	13	
	22			11
$(u_i + v_j)$ addition for non allocated cells from the previous step				

Thus, $\Delta_{ij} = c_{ij} - (u_i + v_j)$ are as shown as below :

As all $\Delta_{ij} > 0$, the optimality criteria is satisfied and the solution is optimal, and cost is minimum. Therefore, optimal solution is $x_{11} = 40$, $x_{12} = 10$, $x_{22} = 60$, $x_{25} = 40$, $x_{31} = 60$, $x_{33} = 50$, $x_{34} = 40$ and minimum transportation cost is Rs. 2700.

		25	26	53
20		25	7	
	13			15

3.9 Variations of a transportation problem :

3.9.1 Maximization Transportation problem :

When, in a TP, the objective is to maximize the total value or benefit, instead of the unit cost c_{ij} , the unit profit or payoff p_{ij} associated with each route is given. The algorithm for solving such problem is same as discussed in section 8.5 except the following two changes :

1. for finding initial solution by VAM method, penalties will be computed as difference between largest and the second largest payoff in the row or column. Allocations are made in those cells where penalty is largest in that row / column.

2. The optimality criteria is that - for all unoccupied cells $\Delta_{ij} \leq 0$.

OR, a maximization TP can be converted into the usual minimization TP by subtracting all the costs from the highest cost involved in the problem. Then our normal methods are applied.

Once we get the optimal solution by either of the above methods, the allocations would be multiplied with the cost entries of the given original problem which was to be maximized and then the total cost/profit is to be found.

3.9.2 Alternative Optimal Solutions :

If an unoccupied cell in an optimal solution has opportunity cost Δ_{ij} equal to zero, then an alternative optimal solution can be formed with another set of allocations without increasing the total transportation cost.

In this case, if such a cell is entered into the basis, no change in transportation cost would occur. We will form a loop for that cell and allocate it the maximum quantity available. After this change, the new solution is obtained and observed.

3.9.3 : Infeasible transportation routes :

When it is not possible to transport goods from certain sources to certain destinations, (say) from source i to sink j, we will assign a very large cost (say) M to the route (i, j). The aim is then, to neglect that route from future consideration.

REVIEW EXERCISE

Solve the following transportation problems:

Q. Sink

		1	2	3	
	1	2	7	4	5
Source	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
		7	9	18	

Ans. $x_{11} = 5$, $x_{22} = 2$, $x_{23} = 6$, $x_{32} = 7$, $x_{41} = 2$, $x_{43} = 12$.

Q. Sink

		1	2	3	4	
	1	10	7	3	6	3
Source	2	1	6	8	3	5
	3	7	4	5	3	7
		3	2	6	4	

Ans. $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$, $x_{33} = 3$, $x_{34} = 2$.

Q. Sink

		1	2	3	4	
	1	19	14	23	11	11
Source	2	15	16	12	21	13
	3	30	25	16	39	19
		6	10	12	15	

Ans. $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 47$, $x_{32} = 7$, $x_{33} = 12$, $z = 796$.

Q. Sink

		1	2	3	4	
	1	1	2	1	4	30
Source	2	3	3	2	1	50
	3	4	2	5	9	20
		20	40	30	10	

Ans. $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{24} = 10$, $x_{32} = 20$.

Q.

	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
	4	2	2	

Ans. $x_{11} = 1$, $x_{21} = 3$, $x_{31} = 0$, $x_{32} = 2$, $x_{33} = 2$, $z = 820$.

Q.

	W_1	W_2	W_3	W_4	
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
	5	8	7	14	

Ans. $x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$, $x_{32} = 6$, $x_{34} = 12$, $z = 743$.

Q.

	A	B	C	D	E	
I	4	1	3	4	4	60
II	2	3	2	2	3	35
III	3	5	2	4	4	40
	22	45	20	18	30	

Ans. $x_{12} = 45$, $x_{15} = 15$, $x_{21} = 17$, $x_{24} = 18$, $x_{31} = 5$, $x_{33} = 20$, $x_{35} = 15$, $z = 290$.

Q.

A	B	C	D

I	15	10	17	18	2
II	16	13	12	13	6
III	12	17	20	11	7
	3	3	4	5	

Ans. $x_{12} = 2$, $x_{22} = 1$, $x_{23} = 4$, $x_{24} = 1$, $x_{31} = 3$, $x_{34} = 4$, $z = 174$.

Q.

	X	Y	Z	
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	

Ans. $x_{13} = 60$, $x_{21} = 50$, $x_{23} = 20$, $x_{32} = 80$, $z = 750$.

Q.

	I	II	III	IV	V	VI	
A	9	12	9	6	9	10	5
B	7	3	7	7	5	5	6
C	6	5	9	11	3	11	2
D	6	8	11	2	2	10	9
	4	4	6	2	4	2	

Ans. $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{33} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$

or $x_{13} = 5$, $x_{22} = 3$, $x_{23} = 1$, $x_{26} = 2$, $x_{31} = 1$, $x_{32} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$, $z = 112$.

Q.

	M_1	M_2	M_3	M_4	
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
	4	3	4	4	

Find the initial basic feasible solution by (1) LCM and (2) VAM. Start with the LCM allocations and get optimal solution.

Ans. $x_{11} = 3$, $x_{23} = 3$, $x_{24} = 4$, $x_{31} = 1$, $x_{32} = 3$, $x_{33} = 1$, $z = 68$.

Q. Determine the optimal solution and the minimum total cost.

22	17	15	30
20	25	19	50
22	26	24	10
50	30	10	

Ans. $x_{12} = 30$, $x_{21} = 40$, $x_{23} = 10$, $x_{31} = 10$, $x_{32} = 0$, $z = 1720$

Q. A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects as shown below :

Project location	Weekly requirements (truck loads)
A	72
B	102
C	41

The company has three gravel plants X, Y and Z located at three different towns. The gravel required by the construction projects can be supplied by these three plants. The amount of gravel which can be supplied by each plant is as follows :

Plant	Amount available / week (truck loads)
X	76
Y	62
Z	77

The company has computed the delivery cost from each plant to each project site. These costs (in rupees) are shown in the following table :

Cost per truck load

		A	B	C
	X	4	8	8
Plant	Y	16	24	16
	Z	8	16	35

a) Schedule the shipment so as to minimize the total transportation cost.

b) Find the minimum cost.

c) Is the solution unique ? If not, find the alternative optimal solution ?

Ans. a) $x_{12} = 76, x_{21} = 21, x_{23} = 41, x_{31} = 51, x_{32} = 56$

b) $z = 2424$

c) No, $x_{12} = 76, x_{22} = 21, x_{23} = 41, x_{31} = 72, x_{32} = 5$.

Q. A company has plants A, B and C which have capacities to produce 300 kg, 200 kg, 500 kg respectively of a particular chemical per day. The production costs (per kg) in these plants are Rs. 70, Rs. 60 and Rs. 66 respectively. Four bulk consumers have placed orders for the product on the following basis :

Consumer	Kg.required / day	Price offered (Rs. / kg.)
I	400	100
II	250	100
III	350	102
IV	150	103

Shipping costs (in rupees per kg) from plants to consumers are given in the following table :

	I	II	III	IV
A	3	5	4	6
B	8	11	9	12
C	4	6	2	8

Find the optimal schedule for the above situation.

Ans. $x_{11} = 150, x_{14} = 150, x_{21} = 200, x_{31} = 50, x_{32} = 100, x_{33} = 350, x_{42} = 150$,

profit = 30700.