INDUSTRIAL STATISTICS AND OPERATIONAL MANAGEMENT

CHAPTER 1: PRE-REQUISITES: VECTORS AND CONVEX SETS

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Keywords

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Vector quantities have both magnitude and direction associated with them. For example, when we speak about velocity, we are not just talking about speed. Velocity is speed in a certain direction. So, the magnitude of the velocity vector might be 50 miles per hour and the direction of the velocity vector might be due north. Vectors are denoted by boldface letters; **u** denotes a vector. When a bold typeface is not available, an arrow is drawn over a normal letter to indicate that it is a. $\vec{\boldsymbol{u}}$ denotes a vector.

1.1 Definitions

A ***scalar*** is a real number. Scalars are denoted by normal letters.

An ***n-vector*** is an ordered sequence of real numbers. For example, $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ is a vector of an ordered n-tuple of real numbers. The real numbers $x_1, x_2, x_3 \ldots x_n$ are called the ***components*** of **u**.

An ***n-dimensional Euclidean space*** is defined to be the collection of n-vectors. The n-dimensional Euclidean space will be denoted by E^n. We shall confine our study to E^n.

For example, 3-dimensional vector $\mathbf{u} = (x_1, x_2, x_3)$ is an ordered triplet of real numbers.

The ***length*** or ***magnitude*** of a vector $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ in E^n is given by

$$\|\mathbf{u}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

Note that, the magnitude of a vector is a scalar. Vertical bars surrounding a boldface letter denote the magnitude of a vector. Since the magnitude is a scalar, it can also be denoted by a normal letter; $u = \|\mathbf{u}\|$ denotes the magnitude of a vector.

Suppose that c is a scalar and $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ is a vector. Then ***scalar multiplication*** of a vector is defined by c**u** $= c(x_1, x_2, x_3 \ldots x_n) = (cx_1, cx_2, cx_3 \ldots cx_n)$. Each component of the vector is multiplied by the scalar. The magnitude of the vector c**u** is c times the magnitude of **u**.

Two vectors with the same dimension can be added to each other or subtracted from each other. The result is another vector.

The ***sum*** of two vectors $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ and $\mathbf{v} = (y_1, y_2, y_3 \ldots y_n)$ is defined to be the vector

$$\mathbf{u} + \mathbf{v} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n).$$

Subtracting **v** from **u** yields

$$\mathbf{u} - \mathbf{v} = (x_1 - y_1, x_2 - y_2, x_3 - y_3 \ldots x_n - y_n).$$

Unit vector is a vector whose magnitude is 1. The standard unit vector in E^n is denoted by e_i in which 1 occurs at i^{th} place and 0 elsewhere. That is, $e_i = (0, 0, \ldots 1, \ldots 0, 0)$.

Let $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ and $\mathbf{v} = (y_1, y_2, y_3 \ldots y_n)$. Then **u** is said to be ***greater than or equal to*** **v** iff $x_k \geq y_k$ for k = 1, 2, …n. This is denoted by writing $\boldsymbol{u} \geq \boldsymbol{v}$.

Note that, two vectors with same dimension can be order comparable.

Suppose we are given two vectors u_1 and u_2. The expressions like $u_1 + u_2$, $u_1 - u_2$, $5u_1 + 7u_2$, $2u_1 - 3u_2$, etc. can be obtained by performing the operations of addition and scalar multiplication. Such vectors are called linear combination of vectors u_1 and u_2. More specifically, the vector $\boldsymbol{v} = \theta_1 \boldsymbol{u_1} + \theta_2 \boldsymbol{u_2}$ is a ***linear combination*** of u_1 and u_2.

Thus, if $\{u_1, u_2, ..., u_p\}$ is a set of p vectors in E^n and $\{\theta_1, \theta_2, ..., \theta_p\}$ is a set of p scalars, then the vector $\boldsymbol{v} = \theta_1 \boldsymbol{u_1} + \theta_2 \boldsymbol{u_2} + ... + \theta_p \boldsymbol{u_p}$ is a ***linear combination*** of the given set of vectors $u_1, u_2, ..., u_p$.

The notion of linear combination provides us following definitions.

Let u_1 and u_2 be two points in E^n. The set $L = \{v | v = \theta u_1 + (1-\theta) u_2\} (\theta \in \square)$ is defined as a ***line*** in E^n passing through $\mathbf{u_1}$ and $\mathbf{u_2}$.

In E^2, with usual notation $\mathbf{v} = (x, y)$ and $\mathbf{u_1} = (x_1, y_1)$ and $\mathbf{u_2} = (x_2, y_2)$, the definition gives

$x = \theta x_1 + (1-\theta) x_2, y = \theta y_1 + (1-\theta) y_2$, that

$$\frac{x - x_2}{x_1 - x_2} = \frac{y - y_2}{y_1 - y_2} = \theta,$$

which is standard form of the equations of the line passing through two points (x_1, y_1) and (x_2, y_2). By restricting θ between 0 and 1 (both inclusive), we get the set of points lying on the ***line segment*** between $\mathbf{u_1}$ and $\mathbf{u_2}$. That is, the set $L = \{v | v = \theta u_1 + (1-\theta) u_2\} (0 \le \theta \le 1)$ is called ***line segment*** joining two points $\mathbf{u_1}$ and $\mathbf{u_2}$. When $\theta = 0$ we are at $\mathbf{u_1}$, when $\theta = 1$ we are at $\mathbf{u_2}$, and when $\theta = 1/2$ we are at the mid-point of the line segment.

1.2 Convex Sets

Convexity is really a simple property to describe. A special class of sets of points in E^n, known as convex sets, plays an important role in optimization theory. Now onwards, unless otherwise stated, a set of vectors or points is always a subset of E^n.

Convex set : A set $S \subseteq E^n$ is said to be ***convex*** if S contains the line segment joining any two points in the set. That is, a set $S \subseteq E^n$ is said to be convex if

$$u_1, u_2 \in S \Rightarrow \theta u_1 + (1-\theta) u_2 \in S; (0 \le \theta \le 1)$$

The above definition can be restated as: A set S is ***convex*** if for any two points u_1 and u_2 belonging to S there are no points on the line between $\mathbf{u}_1$ and $\mathbf{u}_2$ those are not members of S. Equivalently, a set S is convex if there are no points $\mathbf{u}_1$ and $\mathbf{u}_2$ in S such that there is a point on the line between u_1 and u_2 that does not belong to S. The point of this equivalent definition is to include the empty set within the definition of convexity. The definition also includes singleton sets where $\mathbf{u}_1$ and $\mathbf{u}_2$ have to be the same point and thus the line between u_1 and u_2 is the same point.

The set S in figure 1.1 is convex, while the set T is not.

Fig. 1.1 Convex and non – convex sets

Example 1.1 : The unit ball in E^n is convex. The ball is the set $B = \{\mathbf{u} \in E^n : \|\mathbf{u}\| \leq 1\}$.

Suppose that **u** and **v** are in B. Then

$\|\theta \mathbf{u} + (1 - \theta)\mathbf{v}\| \leq |\theta| \|\mathbf{u}\| + |1 - \theta| \|\mathbf{v}\|$

$\leq |\theta| + |1 - \theta| = 1.$

Thus $\theta \mathbf{u} + (1 - \theta)\mathbf{v} \in B$, and thus B is convex.

Example 1.2 : The set E^n is convex. But if we remove any point from E^n, then the resulting set is no longer convex.

Example 1.3 : The set E^n is convex; so is the empty set and a one–point set.

Example 1.4 : Consider the set $S = \{\mathbf{u} \in E^n : \|\mathbf{u}\| = 1\}$, the boundary or surface of unit ball.

Suppose that **u** and **v** be two different points in S. Then

$\|\theta \mathbf{u} + (1 - \theta)\mathbf{v}\| \leq |\theta| \|\mathbf{u}\| + |1 - \theta| \|\mathbf{v}\|$, $0 < \theta < 1$, with equality possible if and only if either (i) $\mathbf{u} = \mathbf{0}$ or (ii) $\mathbf{v} = \mathbf{0}$ or (iii) $\theta \mathbf{u} = t(1 - \theta)\mathbf{v}$, $t > 0$.

Since $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$, neither (i) nor (ii) is true. If (iii) is satisfied, then it is necessary that

$\|\theta \mathbf{u}\| = \| t(1 - \theta)\mathbf{v} \| \Rightarrow \theta \|\mathbf{u}\| = t(1 - \theta)\|\mathbf{v}\| \Rightarrow \mathbf{u} = \mathbf{v}$, which contradict the fact that **u** and **v** are distinct. Hence, (iii) is not true.

Therefore, $\|\theta \mathbf{u} + (1 - \theta)\mathbf{v}\| < 1$. So, S is not convex.

In particular, for n = 2, $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$ and $\theta = 1/2$;

$$\|\theta \mathbf{u} + (1 - \theta)\mathbf{v}\| = \sqrt{1/2} \neq 1$$

Example 1.5 : Any (connected) interval [a, b], (a, b), [a, b), or (a, b] in **R** is convex, as the reader can easily verify, where a and b can be finite or infinite. In fact, the finite interval [a, b] is just the ball of radius (b − a)/2 about (a +

b)/2, which is convex as we saw in Example 2.1. However, if we remove a point from the interior of one of these intervals, the remaining set is no longer convex. In fact, if we remove a point from the interior of any convex set, the resulting set is no longer convex.

For example, $E^n / \{\mathbf{0}\}$ is not convex. The points (1, 0, . . . , 0) and (−1, 0, . . . , 0) are in the set, but the midpoint of the line segment joining these points is not in the set.

Example 1.6 : The solution set S of a system of linear inequalities $A\mathbf{x} = \mathbf{b}$ is convex. For if $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S, then

$$A(\theta \mathbf{x}_1 + (1 - \theta)\mathbf{x}_2) = \theta A\mathbf{x}_1 + (1 - \theta) A\mathbf{x}_2 = \theta \mathbf{b} + (1 - \theta)\mathbf{b} = \mathbf{b}.$$

Sets described as solutions to a system of linear equalities are ***affine sets***.

An important property of convex sets is that intersection of a number of convex sets is convex. As the reader can easily verify, unions and complements of convex sets need not be convex.

Theorem 1.1 : Let $\{S_\alpha\}_{\alpha \in I}$ be a (possibly infinite) collection of convex sets. Then $\bigcap_{\alpha \in I} S_\alpha$ is convex.

Proof : Let **u** and **v** be points in $\bigcap_{\alpha \in I} S_\alpha$, and choose θ between 0 and 1.

Since **u** and **v** are in each S_α , so $\theta \mathbf{u} + (1 - \theta)\mathbf{v} \in S_\alpha$ for each α.

Hence, $\theta \mathbf{u} + (1 - \theta)\mathbf{v} \in \bigcap_{\alpha \in I} S_\alpha$ □

1.3 Constructing a Convex Set :

The defining property of convex sets is that such sets contain the line segment connecting any two points in the set. More generally, choose any finite number, say *P*, of points in a convex set S. Then, the polygon, which has these *P* points as vertices, is contained in S. The following figure illustrates this for a convex set in the plane when $P = 3$ and $P = 4$.

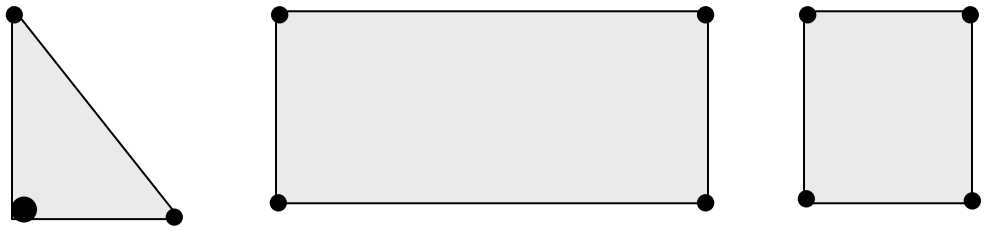


Fig. 1.2 Examples of convex sets

Convex Linear Combination : Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be m points in E^n and let $\theta_1, \theta_2, \ldots \theta_m$ be non-negative real numbers such that $\sum_{j=1}^{m} \theta_j = 1$. Then

$$\mathbf{v} = \theta_1 \mathbf{u}_1 + \theta_2 \mathbf{u}_2 + \ldots + \theta_m \mathbf{u}_m$$

is called the ***convex linear combination*** of points $\mathbf{u}_j$, j = 1, 2,…,m.

For m = 2 this reduce to the definition of a line segment. The definition of a convex set is that the convex combination of any *two* points in the set is contained in the set. In fact a convex set is closed under all ***finite* convex combinations**. We have following theorem in this context.

Theorem 1.2 : A necessary and sufficient condition for a set S to be convex is that every convex linear combination of points in S belongs to S.

Proof : Without loss of generality, we assume that S consists *m* points say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. Suppose that any convex combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is also in S. This implies, any convex combination of two points in S belongs to S. Hence, S is convex. The proof of converse is done by induction on *m*, the number of elements in convex combination. Let S be a convex and $v = \sum_{j=1}^{m} \theta_j u_j$. The Theorem is true for *m* = 1 by default and for *m* = 2 by the definition of convexity. We need only prove the Theorem for *m* = *k* + 1, under the inductive hypothesis that it is true for *m* = *k*.

Suppose that it is true for *m* = *k*.

Now consider $v = \sum_{j=1}^{k+1} \theta_j u_j, \quad \sum_{j=1}^{k+1} \theta_j = 1, \quad \theta_j \geq 0$,*j* = 1, 2, …,*k* + 1

Two cases arise: (i) $\theta_{k+1} = 0$ (ii) $\theta_{k+1} > 0$

Case (i): $\theta_{k+1} = 0 \Rightarrow v = \sum_{j=1}^{k+1} \theta_j u_j = \sum_{j=1}^{k} \theta_j u_j \in S$, by hypothesis.

Case (ii) $\theta_{k+1} > 0 \Rightarrow v = \sum_{j=1}^{k+1} \theta_j u_j$

$$= (1-\theta_{k+1}) \sum_{j=1}^{k} \frac{\theta_j}{(1-\theta_{k+1})} u_j + \theta_{k+1} u_{k+1}$$

Since $\sum_{j=1}^{k+1} \theta_j = 1 \Rightarrow \sum_{j=1}^{k} \theta_j = 1 - \theta_{k+1}$ and so $\sum_{j=1}^{k} \frac{\theta_j}{(1-\theta_{k+1})} = 1$. So the vector $z = \sum_{j=1}^{k} \frac{\theta_j}{(1-\theta_{k+1})} u_j$ is a convex combination of the *k* vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. By the induction hypothesis, **z** is in S. By the definition of convexity

$$v = (1-\theta_{k+1}) z + \theta_{k+1} u_{k+1}$$

is in S.

As we have just seen, a convex set contains all convex combinations of vectors in the set. Similarly, given any collection of vectors, we can identify the convex set that they generate.

Convex hull : Let S be a set of vectors in E^n. The set of all convex combinations of every finite subset of S is called the ***convex hull*** of S and is often written as [S].

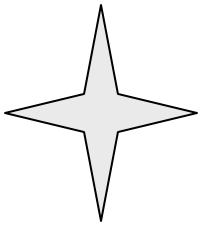
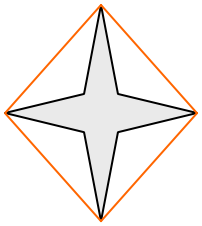
Convex hull of  is 

Fig. 1.3

Clearly, S itself is always contained in [S]. By Theorem 1it follows that if S is convex, then S = [S]. Theorem 1.2 also implies that if S is any set and if T is a convex set containing S, then

$$S \subseteq [S] \subseteq T.$$

This implies that [S] is the smallest convex set containing S; equivalently to say [S] is the "Convex set generated by S". This leads to an alternative definition of [S]: The ***convex hull*** of a set S is the intersection of all convex sets containing S.

Example 1.7 : $S = \{\mathbf{u} \in E^n : \|\mathbf{u}\| = 1\} \Rightarrow [S] = \{\mathbf{u} \in E^n : \|\mathbf{u}\| \leq 1\}$.

Example 1.8 : Let S be a set consisting only two points. Then [S] is the line segment joining given points.

Example 1.9 : For three points the convex hull would be triangle with internal points and the convex hull of the set having 4 point is either any quadrilateral or convex triangle (see figure 1.4) with interior part.

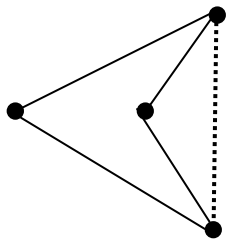


Fig. 1.4

Theorem 1.3 : If $S \in E^n$, each vector in [S] can be written as the convex combination of at most $n + 1$ vectors in S.

We omit the proof of this theorem.

1.4 Hyper planes :

Hyper plane : Choose a vector $\mathbf{a} = (a_1, a_2, a_3 \ldots a_n) \neq \mathbf{0}$ in E^n and a scalar b. The set of the points $\mathbf{u} = (x_1, x_2, x_3 \ldots x_n)$ satisfying $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

i.e.

$$H = \{\mathbf{u} \in E^n: \mathbf{a}.\ \mathbf{u} = b\}$$

is called ***hyper-plane*** in E^n.

In E^2, hyper planes are straight lines. In E^3, hyper planes are planes.

For any n, a hyper plane in E^n divides E^n into two ***half-spaces***, as pictured in Figure 1.5.

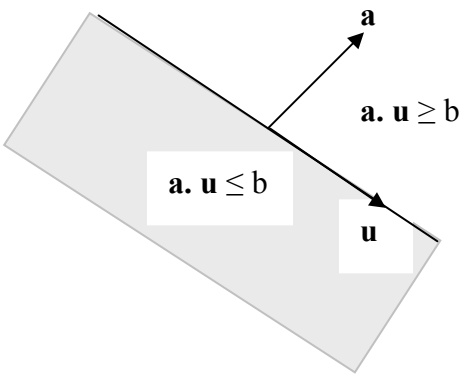


Fig. 1.5

Note : If b = 0, then the hyper plane is said to pass through the origin and its equation is then written as $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.

Closed (Open) half-spaces : The sets $H_- = \{\mathbf{u} \in E^n: \mathbf{a}.\ \mathbf{u} \leq b\}$ and $H_+ = \{\mathbf{u} \in E^n: \mathbf{a}.\ \mathbf{u} \geq b\}$ are called ***closed half-spaces***. If we replace the weak inequalities with strict inequalities, we have ***open half-spaces***. That is, the sets defined by $\{\mathbf{u} \in E^n: \mathbf{a}.\ \mathbf{u} < b\}$ and $\{\mathbf{u} \in E^n: \mathbf{a}.\ \mathbf{u} > b\}$ are called ***open half-spaces***.

It is easy to verify that all hyper planes, open and closed half-spaces are convex.

Polyhedral : A convex set is ***polyhedral*** if it is solution set of finitely many linear inequalities and equalities.

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ denote *m* vectors in E^n, and let $b_1, b_2, \ldots, b_m$ denotes corresponding scalars. Let $H_j = \{\mathbf{u}_j \in E^n: \mathbf{a}_j.\ \mathbf{u}_j \leq b_j\}$. Each H_j is half-space. The set $S = \bigcap_{j=1}^{m} H_j$ is a ***polyhedral convex set***. The set S is set of solution to the system of inequalities:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\le b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\le b_m
\end{aligned}
$$

Bounded convex polyhedra are called convex polytopes. The direction vectors of an unbounded convex polyhedron are vectors in the direction of its bounding rays. Note that a convex polytope has no direction vectors. The hyper planes producing the half spaces are called the ***generating hyper planes*** of the polytope. A polyhedral convex set is drawn in Figure 1.6.

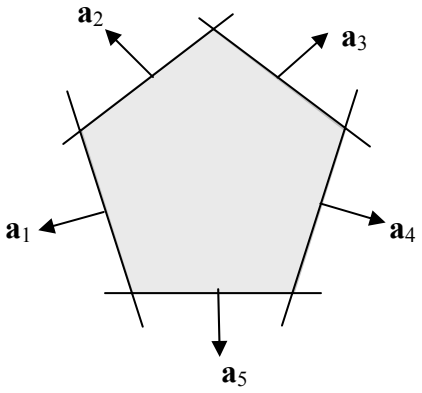


Fig. 1.6

Extreme point of a convex set : An ***extreme point or vertex of a convex set*** is a corner point of polyhedron. More formally, an extreme point is a point which cannot be expressed as a convex combination of other points in the polyhedron.

Thus, a point **u** of a convex set S is an extreme point of the set, if there does not exist any pair of points $\mathbf{u}_1$, $\mathbf{u}_2$ in S, such that

$$\mathbf{u} = \theta\ \mathbf{u}_1 + (1-\theta)\,\mathbf{u}_2 \qquad\qquad 0 < \theta < 1.$$

Note : An extreme point is a boundary point of the set; however, not all boundary points of a convex set are necessarily extreme points.

For example, **a** and **b** are extreme points in the following figure.