

Chapter 6
Renewal Theory and Processes
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Key Words: renewal process, recurrent events, inter-occurrence times, first occurrence, later occurrences, persistent events, transient events, periodic events, delayed recurrent events.

Suggested readings:

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2. Feller, W.(1968), An introduction to Probability Theory and its Applications, Vol. I ,Wiley, New York, 3rd edition.
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6.1 Introduction

Renewal theory deals with repeated trials. In a sequence of repeated trials, several patterns can be defined. As a simple example, a sequence of HT is a pattern in a coin-tossing experiment. A pattern can be recurrent or non-recurrent (transient). A recurrent event (pattern) is that event which occurs repeatedly with probability 1 in an infinitely long sequence of trials while a transient pattern occurs some non-zero p in an infinitely long sequence of trials. Formally, a recurrent event can be defined as follows:

Definition: An integer-valued, or counting process $\{N(t), t \geq 0\}$ corresponding to a series of points distributed in the time interval $[0, \infty)$ is called a renewal process if the inter-occurrence times $T_1, T_2, \dots$ between the successive points are independently and identically distributed random variables.

In case of a Poisson process, we have seen that inter-occurrence times are independently and identically distributed exponential random variables.

Alternatively, a recurrent event may be defined as follows:

Definition: In an infinite sequence of repeated trials, with possible outcomes not necessarily independent, any finite sequence of outcomes, e.g., say $\{E_{j_1}, E_{j_2}, \dots, E_{j_n}\}$ is said to occur if for this sequence $P\{E_{j_1}, E_{j_2}, \dots, E_{j_n}\}$ is defined. A necessary and sufficient condition for this event to be a recurrent event is that

$$P\{E_{j_1}, E_{j_2}, \dots, E_{j_n}, \dots, E_{j_{n+m}}\} = P\{E_{j_1}, E_{j_2}, \dots, E_{j_n}\} P\{E_{j_{n+1}}, \dots, E_{j_{n+m}}\}$$

This definition can be interpreted as two consecutive occurrences of the event in a sequence $\{E_{j_1}, E_{j_2}, \dots, E_{j_n}, \dots, E_{j_{n+m}}\}$ means individual occurrence of the event in two mutually exclusive subsequences $\{E_{j_1}, E_{j_2}, \dots, E_{j_n}\}$ and $\{E_{j_{n+1}}, \dots, E_{j_{n+m}}\}$. This can also be stated as the trials after the occurrence of the event form an exact replica of the experiment and the inter-event times are distributed identically independent of each other.

6.2 Probability distributions associated with the recurrent events

Two basic notions associated with the recurrent events are those of the first occurrence of the event and the later or repeated occurrences of the event. We, now, define probability distributions associated with these two notions.

For a positive integer n , let

$$p_n = P(\text{the event occurs at the } n^{\text{th}} \text{ trial})$$

and

$$f_n = P(\text{the event occurs at the } n^{\text{th}} \text{ trial for the first time})$$

Obviously,

$$p_0 = 1; f_0 = 0$$

Further define the probability generating functions of $\{p_n\}$ and $\{f_n\}$ respectively as

$$P(s) = \sum_{n=1}^{\infty} p_n s^n \text{ and } F(s) = \sum_{n=1}^{\infty} f_n s^n .$$

It is easy to visualize that $\{p_n\}$ is not a proper probability distribution. However, $\{f_n\}$ is a probability distribution subject to whether $f = \sum f_n$ or not. It is easy to see that

$$1 - f = P(\text{ the event does not occur in an infinitely prolonged sequence of trials})$$

Define a random variable T , which is related to the waiting time for the first occurrence of the event, i.e.,

$$P(T = n) = f_n$$

i.e., ω denotes the number of trials needed to have the first occurrence of the event.

Also define a random variable $\tau_n = T_{n+1} - T_n$ as the inter-occurrence time between the n^{th} and the $n+1^{\text{th}}$ occurrence of the event, T_n is the number of trials till the n^{th} occurrence of the event.

Further, let $F_{\tau}(\cdot)$ be the c.d.f. of the i.i.d. random variables τ_n such that

$$F(0) = P(\tau_n = 0) < 1$$

Now, if $f_n^{(2)}$ denotes the probability that the event is occurring for the second time at the n^{th} trial, then

$$\begin{aligned} f_n^{(2)} &= f_1 f_{n-1} + f_2 f_{n-2} + \dots + f_{n-1} f_1 \\ \Rightarrow \{f_n^{(2)}\} &= \{f_n\} * \{f_n\} \\ \Rightarrow F^{(2)}(s) &= (F(s))^2 \end{aligned}$$

Here, $f_n^{(2)}$ denotes the probability distribution of the sum of i.i.d. random variables T_1 and T_2 , both having the probability distribution $P(T_i = n) = f_n; i = 1, 2$

In general, let $f_n^{(r)}$ denotes the probability that the event is occurring for the r^{th} time at the n^{th} trial, and then $\{f_n^{(r)}\}$ denotes the probability distribution of the sum of r i.i.d. random variables $\omega_i, i = 1, 2, \dots, r$ each having the probability distribution $P(T_i = n) = f_n; i = 1, 2, \dots, r$. Then, the g.f. of $\{f_n^{(r)}\}$ is given by $F^{(r)}(s)$.

Also, $\sum_{n=1}^{\infty} f_n^{(r)} = F^{(r)}(1) = f^r$ is the probability that the event occurs at least r times in a sequence of infinitely prolonged sequence of trials and $T_i, i = 1, 2, \dots, r$ denote that the waiting times between successive occurrences of the event.

Renewal function: For a renewal process $\{N(t), t \geq 0\}$, the renewal function $m(t)$ is the expectation of $N(t)$, i.e.,

$$m(t) = E(N(t))$$

If the mean recurrence time between the successive events is μ , then

$$\mu = E(\tau_n) = \int_0^{\infty} t f(t) dt$$

where $f(t)$ is the p.d.f. of $T_n, n = 0, 1, 2, \dots$ such that

$$P(T = n) = f_n$$

Then

$$0 \leq \mu < \infty$$

Define

$$\begin{aligned}\tau^{(r)} &= T_1 + T_2 + \dots + T_r \\ &= \tau_1 + \tau_2 + \dots + \tau_{r-1}\end{aligned}$$

Then $\tau^{(r)}$ is a random variable denoting the time up to the r^{th} occurrence of the event E .

$$\tau^{(0)} = 0$$

Also, we have

$$\begin{aligned}N(t) &= \text{Sup}\{r : \tau^{(r)} \leq t\} \\ N(t) \geq r &\Leftrightarrow r \leq t \\ \Rightarrow P(N(t) = r) &= P(N(t) \geq r) - P(N(t) \geq r - 1) \\ &= P(\tau^{(r)} \leq t) - P(\tau^{(r+1)} \leq t)\end{aligned}$$

where $\tau^{(r)}$ is the r -fold convolution of $F(\cdot)$ with itself.

$$\Rightarrow P(N(t) = r) = F^{(r)}(t) - F^{(r+1)}(t)$$

is the probability distribution of $N(t)$.

Now, we have defined $f = \sum f_n$ and have observed that $\{f_n\}$ is a proper probability distribution subject to whether f is equal to one or not. Depending upon the value of f , the event can be categorized into two categories:

Definition: A recurrent event is called a persistent (recurrent) event if $f = \sum f_n = 1$ and the event is called a transient (recurrent) event if $f = \sum f_n < 1$.

For a persistent event, $\lim_{r \rightarrow \infty} f^r = 1$ and for a transient event $\lim_{r \rightarrow \infty} f^r = 0$. The physical interpretation of these relations is that in an infinitely prolonged sequence of identical trials, a persistent event is bound to occur infinitely often whereas a transient event occurs only a finite number of times and with some non-zero probability $(1-f)$, a transient event may not occur at all.

Periodicity is another property of recurrent events which deals with the regularity with which a recurrent event occurs or recurs:

Definition: A recurrent event is called a periodic recurrent event with period λ if

- (1) $\lambda > 1$
- (2) The event occurs only at trial numbers $\lambda, 2\lambda, 3\lambda, \dots$ and for all other trials $p_n = 0$.
- (3) λ is the greatest integer with the property (2).

In such situations λ is called the period of the event.

6.3 Renewal Theorem: The basic relationship between $\{p_n\}$ and $\{f_n\}$

We have defined

$$p_n = P(\text{the event occurs at the } n^{\text{th}} \text{ trial})$$

and

$$f_n = P(\text{the event occurs at the } n^{\text{th}} \text{ trial for the first time})$$

In order that the event recurs at the n^{th} trial, it is necessary that it occurs for the first time at some intermediate trial ν and then it recurs at the n^{th} trial. The probability of such a formation is $f_\nu p_{n-\nu}$. Since all such formations are mutually exclusive, so

$$\begin{aligned} p_n &= f_1 p_{n-1} + f_2 p_{n-2} + \dots + f_n p_0 \\ \Rightarrow P(s) - p_0 &= P(s)F(s) \\ \Rightarrow P(s) &= \frac{1}{1-F(s)} = \sum_{r=0}^{\infty} (F(s))^r \end{aligned}$$

$$\text{where, } (F(s))^r = F^{(r)}(s) = \sum_{n=0}^{\infty} f_n^{(r)} s^n$$

$$\Rightarrow P(s) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} f_n^{(r)} s^n$$

$$\Rightarrow \sum_{n=0}^{\infty} p_n s^n = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} f_n^{(r)} s^n$$

Equating coefficients of s^n on both the sides, we have

$$p_n = \sum_{r=0}^{\infty} f_n^{(r)}; \quad f_n^{(r)} = 0 \quad \forall \quad r > n$$

This is the renewal theorem for the recurrent events.

The following theorem gives the long run probability of occurrence of a transient event.

Theorem 6.1: for an event to be a transient event, it is necessary and sufficient that $p = \sum p_n$ is finite. In this case the probability f that the event ever occurs is given by

$$f = \frac{p-1}{p}$$

Proof: We have $p = \sum p_n$, $P(s) = \frac{1}{1-F(s)}$; and $f = \sum_n f_n < 1$

Since p_i 's are non-negative so $P(s) \left(= \sum_{n=1}^{\infty} p_n s^n \right)$ is monotonically increasing so,

$$\sum_n^N p_n \leq \lim_{s \rightarrow 1} P(s) \leq \sum_n^{\infty} p_n = p \text{ for every } N$$

And
$$\lim_{s \rightarrow 1} P(s) = \lim_{s \rightarrow 1} \frac{1}{1-F(s)} = \frac{1}{1-F(1)} = \frac{1}{1-f}$$

i.e., if $p < \infty$, then $\frac{1}{1-f} < \infty \Rightarrow f < 1$. Hence the event is transient. Conversely, if the event

is transient, then $f < 1 \Rightarrow p$ is finite.

If the event is persistent then $f = 1 \Rightarrow p = \infty$. Thus the condition is necessary and sufficient. For a transient event

$$\begin{aligned} p &= \sum_n p_n = \frac{1}{1-f} \\ \Rightarrow p(1-f) &= 1 \\ \Rightarrow f &= \frac{p-1}{p} \end{aligned}$$

6.3 Examples

(1) **Nuclear counters:** In a nuclear counter, if particles arriving at the counter follow a Poisson process with intensity $\frac{1}{\lambda}$, then the inter arrival times are distributed as independent exponential variables with mean $\frac{1}{\lambda}$. Thus the arrival of particles is a recurrent event and so is the recording of the particles. We know that an arrival is registered only if the counter is unlocked when arrival takes place. Upon the registration, the counter gets locked for some period. Once the counter is free, then the recording mechanism starts from scratch. Thus the recording process is a renewal process. If an arrival is recorded with probability p , then the generating function of the recurrence times is

$$P(s) = qs + qps^{n+1} + qp^2s^{2n+1} + \dots = \frac{qs}{1 - ps^n}$$

(2) **Successes in Bernoulli trials:** Consider a sequence of independent Bernoulli trials with p as the probability of a success. Then for $n \geq 1$

$$\begin{aligned} p_n &= p \\ \therefore \sum_{n \geq 1} p_n s^n &= p \sum_{n \geq 1} s^n = \frac{ps}{1-s} \\ \Rightarrow P(s) - 1 &= \frac{ps}{1-s} \\ \text{i.e., } P(s) &= \frac{1-qs}{1-s} \text{ and } F(s) = 1 - \frac{1}{P(s)} = \frac{ps}{1-qs} \end{aligned}$$

(3) **Returns to equilibrium:** Consider a particle moving along a straight line. If no outside forces are there, the particle keeps on moving along the straight line. However, some outer forces are working on the particle which keeps on deviating the particle from the position of the equilibrium and the particle keeps on returning to the position of equilibrium. The situation is equivalent to a sequence of independent Bernoulli trials when we want the number of successes equal to the number of failures. The distribution of the particle's return to the equilibrium is a recurrent event. Obviously, return to the equilibrium is possible at an even number of trials, i.e., $k = 2n$. Then

$$\begin{aligned}
p_{2n} &= \binom{2n}{n} p^n q^n = \left(\frac{1}{2} \right) (-4pq)^n \\
\therefore \sum_{n=0}^{\infty} p_{2n} s^{2n} &= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right) (-4pq s^2)^n = \frac{1}{\sqrt{1-4pq s^2}} \\
\Rightarrow P(s) &= \frac{1}{\sqrt{1-4pq s^2}} \\
\text{and } F(s) &= 1 - \frac{1}{P(s)} = 1 - \sqrt{1-4pq s^2}
\end{aligned}$$

To derive the conditions for the event to be persistent or transient, we note that

$$\begin{aligned}
p &= \sum_n p_{2n} = P(1) = \frac{1}{\sqrt{1-4pq}} = \frac{1}{|p-q|} \\
\text{and } f &= \sum_n f_{2n} = F(1) = 1 - |p-q|
\end{aligned}$$

If $u < \infty$, i.e., $p \neq q$, then the event is transient and if $u = \infty$, i.e., $p = q$, then the event is persistent.

(4) **Return to equilibrium through negative values:** Return to equilibrium through negative values occurs if no preceding partial sums are ever positive. The probability distribution of occurrence of this event is then

$$\begin{aligned}
f_{2n}^- &= P(S_{2n} = 0, S_1 < 0, S_2 < 0, \dots, S_{2n-1} < 0) \\
\text{and } f_{2n-1}^- &= 0 \text{ where } S_i = S_1 + S_2 + \dots + S_i
\end{aligned}$$

A return to equilibrium can occur through equally likely events of return through positive values and return through negative values, i.e., $f_{2n}^- = f_{2n}^+ = \frac{1}{2} f_{2n}$

$$\begin{aligned}
\Rightarrow F^-(s) &= \frac{1}{2} F(s) = \frac{1}{2} (1 - \sqrt{1-4pq s^2}) \\
\text{and } P^-(s) &= \frac{1}{1-F^-(s)} = \frac{1 - \sqrt{1-4pq s^2}}{2pq s^2}
\end{aligned}$$

$$\begin{aligned}
f &= \sum_n f_{2n}^- = \frac{1}{2}(1-|p-q|) \\
\sum_n p_{2n} &= P(1) = \frac{1}{|p-q|} \\
\text{and } p &= \frac{1-\sqrt{1-4pq}}{2pq} = \frac{1-|p-q|}{2pq}
\end{aligned}$$

If $p \neq q$, then the event is transient, otherwise the event is persistent.

(5) **Ladder variables:** The partial sums are positive for the first time when the number of successes is more than the number of failures for the first time given that $S_1 < 0$. Also first positive partial sums occur at an odd number of trials, i.e., $k = 2n+1$. The probability distribution of this event is given by

$$\begin{aligned}
f_{2n+1} &= P(S_1 < 0, S_2 < 0, \dots, S_{2n-1} < 0, S_{2n} = 0, S_{2n+1} = 1) \\
&= P(S_1 < 0, S_2 < 0, \dots, S_{2n-1} < 0, S_{2n} = 0, X_{2n+1} = 1)
\end{aligned}$$

$$\text{where } X_n = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q \end{cases}$$

$$\begin{aligned}
\therefore f_{2n+1} &= P(S_1 < 0, S_2 < 0, \dots, S_{2n-1} < 0, S_{2n} = 0) P(X_{2n+1} = 1) \\
&= p p_{2n+1}^-
\end{aligned}$$

Then the generating function $F(s)$ of the first ever-positive partial sums is given by

$$\begin{aligned}
F(s) &= \sum_n f_{2n+1} s^{2n+1} = ps \sum_n p_{2n}^- s^{2n} = ps P^-(s) \\
\Rightarrow F(s) &= ps \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2pqs^2} \right) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}
\end{aligned}$$

The later returns (second and higher ever-positive partial sums) constitute the recurrent events. Let

$$f_{2n}^{(r)} = P(r^{th} \text{ positive sums occur at the } n^{th} \text{ trial})$$

$$\therefore f_{2n}^{(2)} = P(2^{nd} \text{ positive sums occur at the } n^{th} \text{ trial})$$

$$= f_2 f_{n-2} + f_3 f_{n-3} + \dots + f_{n-2} f_2$$

$$= \sum_{m=2}^{n-2} f_m f_{n-m}$$

$$\Rightarrow \{f_{2n}^{(2)}\} = \{f_n\} * \{f_n\}$$

$$\text{or } F^{(2)}(s) = (F(s))^2$$

$$\text{In general, } F^{(k)}(s) = (F(s))^k$$

- (6) **Hazard functions:** Let T is a random variable denoting the time till a system fails. Let $F(x)$ be the c.d.f. of T and $f(x)$ be its p.d.f. Then the hazard function $\rho(x)$ of T is defined as

$$\rho(x) = \frac{f(x)}{1 - F(x)}$$

$$\text{or, } \rho(x)dx = \frac{f(x)dx}{1 - F(x)}$$

i.e., hazard function is the conditional probability that the system will fail in a very small interval

$(x, x + dx)$ given that it has survived up to time x .

$$\text{Then } 1 - F(x) = ce^{\int_0^x \rho(t)dt}$$

$$\text{If } \rho(x) = \lambda(\text{constant}) \text{ for } x > 0 \text{ then}$$

$$F(x) = 1 - \lambda e^{-\lambda x}$$

$$f(x) = \lambda e^{-\lambda x}$$

- (7) **Queuing model:** Consider again an $M/M/1$ queuing system, where customer arrive according to a Bernoulli schedule at time points $1, 2, \dots$; p being the probability of an arrival at any time point. Let X_n is a random variable denoting the service time of the n^{th} customer.

Define an event E , as “the server is free”. Initially, at time point 0, the server is free. A customer arriving the system when the server is free is served immediately thus engaging the server till he is serviced (busy

period). If during this period, any more arrivals are there, they form a waiting line (queue). However, once the server is free after serving all the customers in the queue, the service pattern for future arrivals will be the same as earlier (at $t = 0$). Thus E is a recurrent event.

Let $P(X_n = k) = b_k$, then $B(s) = \sum b_k s^k$ is the generating function of $\{b_k\}$. The generating function $P(s)$ of the new arrivals is

$$P(s) = q + pB(s)$$

and the generating function of the recurrence times is

$$U(s) = qs + psP(s)$$

6.5 Generalized form: Delayed recurrent events

Till now, we have assumed that the probability distributions of the waiting time random variable till the first occurrence of the event and the inter-event time random variables are identical. However, this may not be the situation always and in general waiting time up to the first occurrence of the event has a different distribution than the later inter- recurrence time random variables. Once the event has occurred then the successive waiting time random variables are distributed identically. Such a recurrent event is called a delayed recurrent event where there is a delay in recurrence of the event, which is of same magnitude as the time interval when the event is occurring for the first time. Consider a recurrent event E and let

$$\nu_n = P(E \text{ occurs at the } n^{\text{th}} \text{ trial})$$

Suppose that E occurs for the first time at the k^{th} trial with probability b_k and then recurs in subsequent $n-k$ trials according to probability distribution $\{p_{n-k}\}$, i.e.,

$$\nu_n = b_n + b_{n-1}p_1 + b_{n-2}p_2 + \dots + b_1p_{n-1} + b_0p_n \quad (6.1)$$

i.e., $\{\nu_n\}$ is convolution of $\{b_n\}$ and $\{p_n\}$.

Further, let $V(s) = \sum_{n=0}^{\infty} v_n s^n$; $B(s) = \sum_{n=0}^{\infty} b_n s^n$; and $P(s) = \sum_{n=0}^{\infty} p_n s^n$ be the generating functions

of $\{v_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively. Then, we have

$$V(s) = B(s)P(s) = \frac{B(s)}{1-F(s)} \quad (6.2)$$

The following theorem gives the probability $\{v_n\}$ of a delayed recurrent event and specifies the condition when this will be a proper probability distribution.

Theorem 6.2: If $p_n \rightarrow a$, then $v_n \rightarrow ab$,

where $b = B(1) = \sum_n b_n$.

If $\sum_n p_n \rightarrow \alpha$, then $\sum_n v_n \rightarrow \alpha b$.

Proof: Denote by

$$\begin{aligned} \eta_k &= P(\text{first renewal period is larger than } k) \\ &= b_{k+1} + b_{k+2} + \dots \end{aligned}$$

Let k is sufficiently large so that for a pre assigned number ε , $\eta_k < \varepsilon$.

Now, from (6.1), we have

$$\begin{aligned} b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} &\leq v_n = b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} \\ &\quad + \{b_{k+1} p_{n-(k+1)} + \dots + b_{n-1} p_1 + b_n\} \\ &\leq b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} + \{b_{k+1} + \dots + b_{n-1} + b_n\} \\ &\leq b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} + \eta_k \end{aligned}$$

As $p_n \rightarrow a$ so

$$\begin{aligned}
b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} &\rightarrow \{b_0 + b_1 \dots + b_k\} a \\
&= (b - \eta_k) a \\
&> ba - \varepsilon a \\
&> ba - 2\varepsilon \text{ as } a < 2
\end{aligned}$$

Also,

$$\begin{aligned}
b_0 p_n + b_1 p_{n-1} + \dots + b_k p_{n-k} + \eta_k &\rightarrow (b - \eta_k) a + \eta_k \\
&= ba + \eta_k (1 - a) \\
&< ba + (1 - a) \varepsilon \\
&< ba + 2\varepsilon
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow ba - 2\varepsilon < \nu_n < ba + 2\varepsilon \\
\text{or, } \lim_{n \rightarrow \infty} \nu_n &\rightarrow ab
\end{aligned}$$

Again, $\sum_n \nu_n = V(1) = B(1)P(1) \rightarrow \alpha b$ as $P(1) = \sum_n p_n \rightarrow \alpha$.

Hence the result.

Corollary: If the event E is a persistent event, then

$$\begin{aligned}
\nu_n &\rightarrow \frac{b}{\mu} \text{ as } \nu_n = \lim_{s \rightarrow 1^-} (1-s)V(s) \\
&= \lim_{s \rightarrow 1^-} B(s) \frac{(1-s)}{(1-F(s))} \\
&\rightarrow \frac{B(1)}{F'(1)} = \frac{b}{\mu}
\end{aligned}$$

Theorem 6.3: (Renewal theorem): Let $\{f_n, n = 1, 2, 3, \dots\}$ and $\{b_n, n = 1, 2, 3, \dots\}$ are two sequences of real numbers such that

$$\begin{aligned}
f_n &\geq 0; \quad f = \sum_n f_n < \infty \text{ and} \\
b_n &\geq 0; \quad b = \sum_n b_n < \infty
\end{aligned} \tag{6.3}$$

Define a sequence $\{v_n, n = 0, 1, 2, 3, \dots\}$ by the convolution relation

$$v_n = b_n + b_{n-1}f_1 + b_{n-2}f_2 + \dots + b_1f_{n-1} + b_0f_n \quad (6.4)$$

i.e.,
$$v_n = b_n + \sum_{r=1}^n f_r b_{n-r}$$

Further let $\{f_n\}$ is not periodic. Then

(i) If $f < 1$, then $v_n \rightarrow 0$ and $\sum_n v_n = \frac{b}{1-f}$

(ii) If $f = 1$, then $v_n \rightarrow \frac{b}{\mu}$.

Proof: From (6.4), we have

$$V(s) = \frac{B(s)}{1-F(s)} \quad (6.5)$$

(i) If $f < 1$, then from (6.5)

$$V(1) = \frac{B(1)}{1-f}$$

or,
$$\sum_n v_n = \frac{b}{1-f} < \infty$$

$$\Rightarrow v_n \rightarrow 0$$

(ii) If $f = 1$, then $\sum_n v_n = \frac{b}{1-f} \rightarrow \infty$

As $\mu = F'(1) = \sum_n n f_n$, so $v_n \rightarrow \frac{b}{\mu}$ (see corollary)

6.6 Examples

- (1) **System reliability:** Consider a system consisting of n independent components, which may fail individually. The system, however, remains functional till at least one component is in working condition at any time point. If all the components fail, then the system also fails. Let the component i ($i = 1, 2, \dots, n$) remains functional for a time-period which is exponential with mean λ_i . If it fails, it remains in that state for a time period, which is exponential with mean μ_i . A failed component is repaired, after

which it becomes functional again. Under the assumption that repeated failures do not deteriorate the component otherwise, the number $N(t)$ of failures in $[0, t]$ constitutes a delayed recurrent event. However, if there is any deterioration in the components as a result of failures or otherwise, subsequent failures will become more and more probable and the event will not be a recurrent event as after repair, it is not a replica of the original event.

- (2) **Replacement policy:** In any system, which functions by the simultaneous working of several components, each of which is exposed to the risk of failure, some type of replacement policy is required. This replacement policy may be in addition to the repair policy. If a failed component may not be repaired, then it has to be replaced and, in general, the usual replacement policy is to replace the failed component by a similar one.

However, if a system fails as a result of component failure, it may lead to heavy losses. In order to avoid such a situation, other replacement policies may be adopted. One of such policies is age-replacement policy where a component is replaced immediately on failure or when it has attained a specified age, whichever is earlier. Another policy is block replacement policy where a component is replaced immediately on failure and in addition, at specified periods, all the components, even if they are working, are replaced. If $N(t)$ is the number of failures in $[0, t]$, then $\{N(t), t \geq 0\}$ constitutes a delayed recurrent event.

- (3) In a delayed recurrent event, let the first occurrence time is distributed hyper-exponentially and the later times are ordinary exponential variables with parameter λ . Then

$$f_{X_1}(t) = p\gamma e^{-\gamma t} + (1-p)\theta e^{-\theta t}, \quad 0 \leq p \leq 1; \quad \lambda > \theta > 0$$

In this case, p is the proportion of the components having high failure rate γ and $1-p$ is the proportion of components having the low failure rate θ .

For X_1 , the renewal function $m_{X_1}(t)$ is given by

$$m_{X_1}(t) = \int_0^t f_{X_1}(t) dt = \frac{p(\gamma - \theta)}{\gamma} (1 - e^{-\gamma t})$$

and for the later occurrence the renewal function $m(t)$ is given by

$$m(t) = \theta t$$

Then the renewal function of the delayed recurrent event $m_D(t)$ is

$$m_D(t) = \theta t + \frac{p(\gamma - \theta)}{\gamma} (1 - e^{-\gamma t})$$

(4) **Nuclear counters:** In the renewal process $\{M(t), t \geq 0\}$ of the recording of the arriving particles, let

$\tau_1 = T_1$ is the time till the first recording. The subsequent inter-recording times are

$$\tau_i = T_{i+1} - T_i; \quad i = 1, 2, 3, \dots$$

Then the characteristic function of the i.i.d. inter-recording times is given by

$$\phi_\tau(s) = \left(1 - \frac{is}{\lambda} e^{(\lambda - is)L} \right)^{-1}$$

where L is the locking period of the counter.

$$E(\tau) = \frac{1}{\lambda} e^{\lambda L}$$

$$\text{Var}(\tau) = \frac{e^{2\lambda L}}{\lambda^2} - \frac{2Le^{\lambda L}}{\lambda}$$

6.7 The limiting behaviour of recurrent events

In case of persistent recurrent events, we have seen that the probability of recurrence tends to infinity. However, under certain regulatory conditions, the asymptotic behaviour of the persistent recurrent events follows a normal distribution. We are stating the following result without proof:

Theorem 6.4 (Normal approximation theorem): Let the recurrent event E is persistent and its recurrence times have finite mean μ and variance σ^2 . Then the waiting time random variable $T^{(n)}$ denoting the time till the r^{th} occurrence of E and the number N_n of occurrences of E in the first n trials are asymptotically normally distributed.

Problems

1. Find the distribution functions corresponding to the following hazard functions:

(i) $\rho(x) = ae^{-bx}; \quad a, b > 0$

(ii) $\rho(x) = \frac{b}{x+a}; \quad a, b > 0$

2. Show that

$$P(N(t) = r) = \frac{1}{\mu} \int_0^t (P(N(u) = r-1) - P(N(u) = r)) du$$

3. Find the second moments of the renewal process having the first renewal interval as hyper exponential variable and later intervals as ordinary exponential variables.