

Chapter 5
Branching Processes
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Key words: Branching processes, ancestors, offspring, generations, generating function, and ultimate extinction.

Suggested readings:

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5.1 Introduction

The branching processes are those stochastic processes, which deal with specialized populations of those objects, which can reproduce objects of the similar kind. The objects can be living biological entities, such as human population, animal population, plants, cells, genes or viruses etc. or the objects can be non-living physical entities such as radio-active particles or nuclear matter. The study of branching processes entails the behaviour of such populations over different generations. The branching processes may be discrete or continuous in time. In continuous time branching processes, the process of reproduction is continuous over time and it may be difficult to identify different generations. However, in discrete time branching processes, different generations can be identified distinctly. In this chapter, we will study discrete time branching processes.

We define two basic terms associated with branching processes:

Ancestors: The initial set of population with which the process starts are called the ancestors or the 0th generation.

These ancestors reproduce to create objects of the similar type, which form the first generation, the direct descendents of the ancestors. It is assumed that each reproduction takes place independently, i.e., there is no interference in the process of reproduction. The process goes on to generate 2nd, 3rd and subsequently n^{th} generation.

Progeny: The descendents (offspring) of ancestors are called the progeny.

The number of objects in any generation is a random variable, which depends upon the size of the previous generation (Markovian property)

We, now, define a branching process, which in general is known as a G.W. branching process (after the names of Galton and Watson, who proposed this notion for the first time).

Branching process: Let the size of the i^{th} generation be represented by the random variable X_i ($i = 0, 1, 2, \dots$).

Let

$p_k = P(\text{an object generates } k \text{ similar objects, irrespective of the generation to which it belongs})$

$$p_k \geq 0; \quad k = 0, 1, 2, \dots; \quad \sum_k p_k = 1$$

Then $\{X_i, i = 0, 1, 2, \dots\}$ is a family of random variables such that each member of X_i reproduces according to the reproduction schedule determined by the offspring distribution $\left\{ p_k; p_k \geq 0 \quad \forall \quad k; \sum_k p_k = 1 \right\}$. The sequence $\{X_i, i = 0, 1, 2, \dots\}$ is called a (G.W.) branching process with offspring distribution $\{ p_k \}$.

Without any loss of generality, it can be assumed that $X_0 = 1$. It simply means an object in the population is the minimum number of units necessary for reproduction. It could be a single unit as in the case of nuclear fission or two units as in the case of human reproduction. As we have seen earlier also, the size of any generation depends on the size of the previous generation. Then the conditional probability that the size of the $n+1^{\text{th}}$ generation is k given that the size of the n^{th} generation is j , is given by

$$p_{jk} = P(X_{n+1} = k | X_n = j); \quad j, k = 0, 1, 2, \dots$$

Thus given the initial condition $X_0 = 1$, $\{X_i, i = 0, 1, 2, \dots\}$ is a Markov chain with transition probabilities $\{p_{jk}; j, k = 0, 1, 2, \dots\}$, i.e., the transition probability matrix has infinite order (if the population does not become extinct at any point of time).

Examples:

1. In genetics, it has been found that each gene can reproduce k ($k = 1, 2, \dots$) identical offspring, each of which has a non-zero probability of getting transformed, which is, then, called a mutant gene. This mutant gene is, the ancestor for a generation of genes of identical type. Let X_0 be the initial size of the population and p_k be the probability that an individual gene reproduces k identical genes. Then

$$\left\{ p_k; p_k \geq 0 \quad \forall \quad k; \sum_k p_k = 1 \right\}$$

is the probability distribution of $\{p_k\}$. If X_n be the size of the population at the n^{th} generation, then $\{X_n, n \geq 1\}$ is a branching process.

2. In nuclear chain reactions, a nucleus is split by colliding with a neutron. This process is called a **fission** process. The process generates a random number of new neutrons of the identical type which are called the secondary neutron. These secondary neutrons, again, participate in a fission process, thus generating a new

generation of neutron. A vast amount of energy is produced in such chain reactions and if the process is uncontrolled, then it may cause a blast (nuclear bomb). The size of the generations of new neutrons is a branching process.

5.2 Generating functions of a branching process

Generating functions play a major role in the study of branching processes. Two generating functions associated with the processes are generating function of the offspring distribution and the generating function of the random variable denoting the size of a generation. We now define these generating functions and obtain a relationship between the two.

Let ξ_r is the number of offspring reproduced by the r^{th} member of the n^{th} generation. Since offspring distribution is independent of the location of the generation on the time axis, it remains the same for all values of n . Then the size of the $n+1^{\text{th}}$ generation is given by

$$X_{n+1} = \xi_1 + \xi_2 + \dots + \xi_{X_n} \quad (5.1)$$

where X_n is the size of the n^{th} generation and is a random variable.

Then ξ_r , $r = 1, 2, \dots$ are i.i.d. random variables with the probability distribution $\{p_k\}$.

(5.1) is a characterization of the G.W. branching process and can be taken as an alternative definition.

Let the probability generating function of the offspring distribution $\{p_k\}$, denoted by $\psi(s)$ is defined as

$$\psi(s) = \sum_k P(\xi_r = k) s^k = \sum_k p_k s^k \quad (5.2)$$

Further, let the probability generating function of the branching process $\{X_i, i = 0, 1, 2, \dots\}$ is given by

$$\psi_n(s) = \sum_k P(X_n = k) s^k; \quad n = 0, 1, 2, \dots \quad (5.3)$$

Under the initial condition $X_0 = 1$, we prove the following characterization theorem for a branching process:

Theorem 5.1: For a G.W. branching process,

$$\psi_n(s) = \psi_{n-1}(\psi(s)) = \psi(\psi_{n-1}(s)); \quad n = 0, 1, 2, \dots \quad (5.4)$$

Proof: For $n = 0$,

$$\psi_0(s) = \sum_k P(X_0 = k) s^k = s$$

For $n = 1$,

$$\begin{aligned} \psi_1(s) &= \sum_k P(X_1 = k) s^k = \sum_k \left(\sum_j^{X_0} P(\xi_j = k) \right) s^k \\ &= \sum_k P(\xi_1 = k) s^k = \psi(s) \end{aligned}$$

Let the result stands for $n = m$, where m is a positive integer, i.e.

$$\psi_m(s) = \psi_{m-1}(\psi(s))$$

We have

$$\begin{aligned} P(X_m = k) &= \sum_{i=0}^{\infty} P(X_m = k | X_{m-1} = i) P(X_{m-1} = i) \\ &= \sum_{i=0}^{\infty} P\left(\sum_{j=1}^i \xi_j = k\right) P(X_{m-1} = i) \end{aligned}$$

Then, for different values of n , we have

$$\begin{aligned} \psi_n(s) &= \sum_{k=0}^{\infty} P(X_n = k) s^k \\ &= \sum_{k=0}^{\infty} s^k \left(\sum_{i=0}^{\infty} P\left(\sum_{j=1}^i \xi_j = k\right) P(X_{n-1} = i) \right) \\ &= \sum_{i=0}^{\infty} P(X_{n-1} = i) \left(\sum_{k=0}^{\infty} P\left(\sum_{j=1}^i \xi_j = k\right) s^k \right) \end{aligned}$$

where, $\sum_{k=0}^{\infty} P\left(\sum_{j=1}^i \xi_j = k\right) s^k$ is the probability generating function of i i.i.d. random variables ξ_j each

having the p.g.f. $\psi(s)$

$$\begin{aligned} \therefore \sum_{k=0}^{\infty} P\left(\sum_{j=1}^i \xi_j = k\right) s^k &= (\psi(s))^i \\ \Rightarrow \psi_n(s) &= \sum_{i=0}^{\infty} P(X_{n-1} = i) (\psi(s))^i = \psi_{n-1}(\psi(s)) \end{aligned}$$

Now

$$\begin{aligned}
\psi_n(s) &= \psi_{n-1}(\psi(s)) \\
&= \psi_{n-2}(\psi(\psi(s))) \\
&= \psi_{n-2}(\psi^2(s)) \\
&= \psi_{n-3}(\psi(\psi^2(s))) \\
&= \psi_{n-3}(\psi^3(s)) = \cdots \psi_{n-k}(\psi^k(s)); \quad k = 0, 1, 2, \dots, n
\end{aligned}$$

Putting $k = n-1$, we have

$$\psi_n(s) = \psi(\psi_{n-1}(s))$$

Putting $k = 1$, we have

$$\psi_n(s) = \psi_{n-1}(\psi(s))$$

Thus

$$\psi_n(s) = \psi_{n-1}(\psi(s)) = \psi(\psi_{n-1}(s))$$

The above result has an interesting generalization. Even if we consider the process of reproduction after every r^{th} generation, still the resultant process is a branching process.

Theorem 5.2: If $\{X_i, i = 0, 1, 2, \dots, X_0 = 1\}$ is a G.W. branching process, with the offspring distribution

$\left\{ p_k; p_k \geq 0 \forall k; \sum_k p_k = 1 \right\}$, then $\{X_{ri}, i = 0, 1, \dots, 2; X_0 = 1\}$, where r is a fixed positive integer is also a

branching process.

Proof: Let $\psi(s) = \sum_k P(\xi_r = k) s^k = \sum_k p_k s^k$ be the offspring p.g.f. of $\{X_i, i = 0, 1, 2, \dots, X_0 = 1\}$ where

$$p_k = P(\text{an object generates } k \text{ similar objects, irrespective of the generation to which it belongs})$$

Now $\{X_{ri}, i = 0, 1, \dots, 2; X_0 = 1\}$ for fixed r is a stochastic process on $0^{\text{th}}, r^{\text{th}}, 2r^{\text{th}}, \dots$ generations.

For $r = 1$, the process coincides with the parent process.

Define

$$p_{rk} = P(\text{an object generates } k \text{ similar objects, after every } r \text{ generations})$$

and

$$\psi_r(s) = \sum_k p_{rk} s^k$$

Further, let $\psi_{ri}(s) = \sum_j P(X_{ri} = j) s^j$ be the p.g.f. of $\{X_{ri}, i = 0, 1, \dots; X_0 = 1\}$, i.e., the size of the population at the ri^{th} generation. Then,

$$\begin{aligned} P(X_{ri} = j) &= \sum_{l=0}^{\infty} P(X_{ri} = j | X_{r(i-1)} = l) P(X_{r(i-1)} = l) \\ &= \sum_{l=0}^{\infty} P\left(\sum_{m=1}^l \xi_{rm} = j\right) P(X_{r(i-1)} = l) \end{aligned}$$

$$\text{where, } X_{ri} = \sum_{m=1}^{X_{r(i-1)}} \xi_{rm}$$

$$\begin{aligned} \therefore \psi_{ri}(s) &= \sum_{j=0}^{\infty} P(X_{ri} = j) s^j \\ &= \sum_{j=0}^{\infty} s^j \left(\sum_{l=0}^{\infty} P\left(\sum_{m=1}^l \xi_{mr} = j\right) P(X_{r(i-1)} = l) \right) \\ &= \sum_{l=0}^{\infty} P(X_{r(i-1)} = l) s^l \sum_{j=0}^{\infty} s^j P\left(\sum_{m=1}^l \xi_{mr} = j\right) \\ &= \sum_{l=0}^{\infty} P(X_{r(i-1)} = l) (\psi_r(s))^l \\ &= \psi_{r(i-1)}(\psi_r(s)) \end{aligned}$$

Thus $\psi_{ri}(s)$ is a p.g.f. associated with a branching process. Hence $\{X_{ri}, i = 0, 1, \dots; X_0 = 1\}$ is a branching process.

Moments of X_n

Now, we obtain the moments of the random variable X_n .

Define

$$\begin{aligned} \psi'(1) &= \sum_k k P(\xi_j = k) \\ &= E(\xi_j) = E(X_1) = \mu(\text{say}) \end{aligned}$$

and

$$\sigma^2 = \psi''(1) + \psi'(1) - (\psi'(1))^2$$

Then, we obtain expressions for μ and σ^2 .

Theorem 5.3: For a G.W. branching process

$$E(X_n) = \mu^n; \text{ and}$$

$$\text{Var}(X_n) = \begin{cases} \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1} \sigma^2, & \text{if } \mu \neq 1 \\ n\sigma^2, & \text{if } \mu = 1 \end{cases}$$

Proof: Differentiating (5.4) w.r.t. s at $s = 1$, we have

$$\begin{aligned} \psi_n'(s) \Big|_{s=1} &= \psi_{n-1}'(\psi(s)) \psi'(s) \Big|_{s=1} \\ &= \mu \psi_{n-1}'(1) \\ &= \mu^2 \psi_{n-2}'(1) \\ &= \dots \\ &= \mu^{n-1} \psi'(1) \\ &= \mu^n \end{aligned}$$

$$\Rightarrow E(X_n) = \mu^n$$

Now,

$$\begin{aligned} \psi_n''(s) \Big|_{s=1} &= \psi_{n-1}''(\psi(s)) (\psi'(s))^2 \Big|_{s=1} + \psi_{n-1}'(\psi(s)) \psi''(s) \Big|_{s=1} \\ &= \mu^2 \psi_{n-1}''(1) + \psi_{n-1}'(1) \psi''(1) \\ &= \mu^2 \psi_{n-1}''(1) + \mu^{n-1} \psi''(1) \end{aligned}$$

$$\Rightarrow \text{Var}(X_n) = \psi_n''(1) + \mu^n - \mu^{2n}$$

and

$$\begin{aligned}
\psi_n''(1) &= \mu^2 \psi_{n-1}''(1) + \mu^{n-1} \psi''(1) \\
&= \mu^2 (\mu^2 \psi_{n-2}''(1) + \mu^{n-2} \psi''(1)) + \mu^{n-1} \psi''(1) \\
&= \mu^4 \psi_{n-2}''(1) + \mu^{n-1} (\mu+1) \psi''(1) \\
&= \mu^4 \psi_{n-2}''(1) + \mu^{n-1} \frac{(\mu^2-1)}{(\mu-1)} \psi''(1) \\
&= \dots \\
&= \mu^{2(n-1)} \psi_{n-(n-1)}''(1) + \mu^{n-1} \frac{(\mu^{n-1}-1)}{(\mu-1)} \psi''(1) \\
&= \mu^{n-1} \left(\mu^{n-1} + \frac{\mu^{n-1}-1}{\mu-1} \right) \psi''(1) \\
&= \mu^{n-1} \left(\frac{\mu^n-1}{\mu-1} \right) \psi''(1)
\end{aligned}$$

$$\begin{aligned}
\text{where, } \psi''(1) &= \text{Var}(X_1) - \psi'(1) + (\psi'(1))^2 \\
&= \sigma^2 - \mu + \mu^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \text{Var}(X_n) &= \mu^{n-1} \left(\frac{\mu^n-1}{\mu-1} \right) (\sigma^2 - \mu + \mu^2) + \mu^n - \mu^{2n} \\
&= \mu^{n-1} \left(\frac{\mu^n-1}{\mu-1} \right) \sigma^2 + \mu^n (\mu^n - 1) + \mu^n - \mu^{2n} \\
&= \mu^{n-1} \left(\frac{\mu^n-1}{\mu-1} \right) \sigma^2; \text{ if } \mu \neq 1
\end{aligned}$$

$$\text{If } \mu = 1, \text{ then } \psi_n'(s) \Big|_{s=1} = 1$$

and

$$\begin{aligned}
\psi_n''(s) \Big|_{s=1} &= \psi_{n-1}''(1) + \psi''(1) \\
&= \psi_{n-2}''(1) + 2\psi''(1) \\
&= \dots
\end{aligned}$$

$$\begin{aligned}
&= n\psi''(1) \\
&= n\left(\sigma^2 + \psi'(1) - (\psi'(1))^2\right) \\
&= n\sigma^2
\end{aligned}$$

Hence the result.

Probability of extinction: If for some value of n , say, $n = n_1$, $P(X_{n_1} = 0) = 1$, then the population becomes extinct at the n_1^{th} generation and $P(X_{n_1+1} = 0 \mid X_{n_1} = 0) = 1$.

We state and prove the following results.

Theorem 5.4: If $\mu \leq 1$, then the probability of ultimate extinction is 1. If $\mu > 1$, the probability of ultimate extinction is the positive root less than unity, of the equation $s = \psi(s)$.

Proof: Define q_n as the probability of ultimate extinction at the n^{th} generation, i.e., $q_n = P(X_n = 0)$

$$\begin{aligned}
\psi_n(s) &= \sum_{k=0}^{\infty} P(X_n = k) s^k \\
&= P(X_n = 0) + \sum_{k=1}^{\infty} P(X_n = k) s^k \\
&= q_n + \sum_{k=1}^{\infty} P(X_n = k) s^k \\
\Rightarrow q_n &= \psi_n(0) \\
q_1 &= \psi_1(0) = \psi(0) = p_0
\end{aligned}$$

Now,

$$\begin{aligned}
\psi_n(s) &= \psi(\psi_{n-1}(s)) \\
\Rightarrow q_n &= \psi(q_{n-1})
\end{aligned} \tag{5.5}$$

Now, $p_0 = P(\text{an object generates no similar objects})$

So if $p_0 = 0 \Rightarrow q_1 = 0$, i.e., extinction can never occur since the probability of no offspring is zero.

If $p_0 = 1 \Rightarrow q_1 = 1, q_2 = 1, \dots$, i.e. the probability of no offspring is one and the extinction is certain to occur right after the 0th generation.

Consider the case when $0 < p_0 < 1$. Now $\psi(s) = \sum_k p_k s^k$ is a strictly increasing function of s so,

$$\begin{aligned} q_2 &= \psi(q_1) = \psi(p_0) > \psi(0) = q_1 \\ &\Rightarrow q_2 > q_1 \end{aligned}$$

Now, assume that $q_n > q_{n-1}$. So

$$\begin{aligned} q_{n+1} &= \psi(q_n) > \psi(q_{n-1}) = q_n \\ &\Rightarrow q_{n+1} > q_n \end{aligned}$$

Thus by mathematical induction $q_1 < q_2 < q_3 < \dots$ and this monotonically increasing sequence of $\{q_n\}$ is bounded above by 1, so there must exist an upper limit for the sequence, i.e., $\lim_{n \rightarrow \infty} q_n = q$ (say); $0 \leq q \leq 1$, which is the probability of ultimate extinction, and from (5.5), we find that q satisfies the equation $q = \psi(q)$, i.e., q is a root of the equation

$$s = \psi(s) \tag{5.6}$$

Now we show that q is the smallest positive root of the equation (5.6).

Let s_0 be any other arbitrary positive root of (5.6). Then

$$q_1 = \psi(0) < \psi(s_0) = s_0.$$

Assume that $q_m < s_0$, then $q_{m+1} = \psi(q_m) < \psi(s_0) = s_0$. Hence, by induction

$$\begin{aligned} q_n &< s_0 \quad \forall \quad n \\ &\Rightarrow q = \lim_{n \rightarrow \infty} q_n \leq s_0 \end{aligned}$$

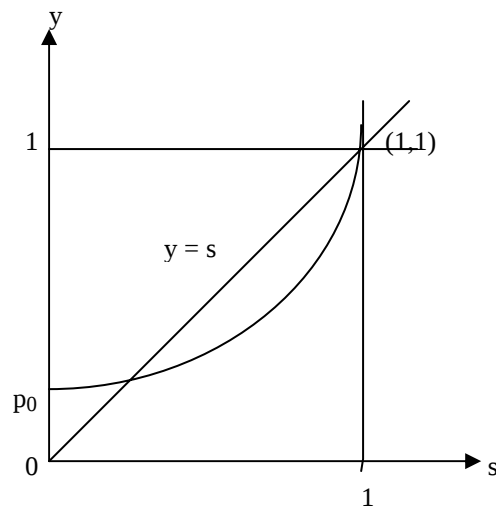
Hence q is the smallest positive root of the equation (5.6).

Now, consider the graph of $y = \psi(s)$ in $0 \leq s \leq 1$.

It starts with the point $(0, p_0)$ and ends at the point $(1, 1)$. As $P(s)$ is increasing function of s , so the curve is a convex curve, lying entirely in the first quadrant and intersecting the line $y = s$ in at most two points. So (5.6) can have at most two roots one of which is unity. We identify two cases:

Case I: The curve $y = \psi(s)$ lies entirely above the line $y = s$.

****Fig.(i)



(file graph_ii.doc)

Then $(1, 1)$ is the only point of intersection, i.e., unity is the only root of the equation (5.6) and

$q = \lim_{n \rightarrow \infty} q_n = 1$. Then

$$\psi(1) - \psi(s) = 1 - \psi(s) \leq 1 - s \quad \text{as } \psi(s) > s \text{ above the line } y = s$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\psi(1) - \psi(s)}{1 - s} \leq 1$$

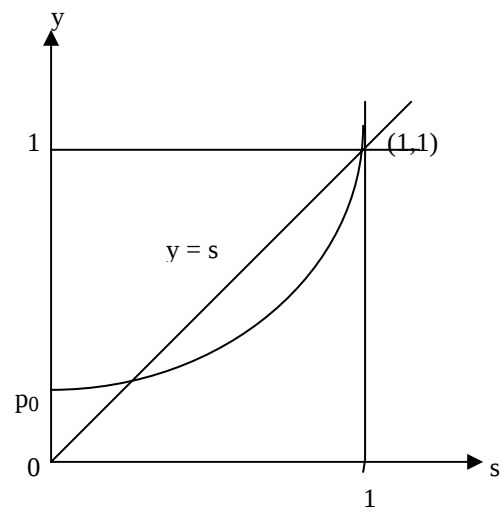
$$\text{or, } \psi'(1) \leq 1 \text{ i.e., } \mu \leq 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} q_n = 1 \text{ when } \mu \leq 1.$$

Case II: The curve $y = \psi(s)$ intersects the line $y = s$ at another point $(\delta, \psi(\delta))$ such that

$\delta = \psi(\delta)$, $\delta < 1 \Rightarrow$ there is another root of (5.6), i.e., $\delta (< 1)$.

Fig.(ii)



(file graph_i.doc)

Since the curve $y = \psi(s)$ is convex, it lies below the line $y = s$ in $(\delta, 1)$ and above the line $y = s$ in $(0, \delta)$, i.e.,

$$\begin{aligned}\psi(s) &< s \text{ when } \delta < s < 1 \\ \text{and } \psi(s) &> s \text{ when } 0 < s < \delta \\ \Rightarrow q_1 = \psi(0) &< \psi(\delta) = \delta\end{aligned}$$

Assume that $q_m < \delta \Rightarrow q_{m+1} = \psi(q_m) < \psi(\delta) = \delta$

So by induction, we have

$$\begin{aligned}q_m &< q_{m+1} < q_{m+2} < \dots < \delta \\ \Rightarrow \lim_{n \rightarrow \infty} q_n &= \delta (< 1) \\ \text{or, } q &= \delta (< 1)\end{aligned}$$

Now, consider the interval $[\delta, 1]$. By Lagrange's mean-value theorem, there exists a value ξ in $0 < \xi < 1$ such that

$$\psi'(\xi) = \frac{\psi(1) - \psi(\xi)}{1 - \xi} = 1$$

Since the derivatives are also monotonic increasing so

$$\begin{aligned}\psi'(1) &> \psi'(\xi) = 1 \\ \text{i.e., } \mu &> 1\end{aligned}$$

So q is the root of (5.6), less than unity, corresponding to $\mu > 1$. Hence the result.

Remark: We have shown in the above theorem that q satisfies equation (5.6), it being an upper limit of the sequence of probabilities. The same can be shown using an independent argument also.

Assume that the population starts with k individuals in the beginning, i.e. $X_0 = k$. Then

$$\begin{aligned}q &= P(\text{ultimate extinction}) \\ &= \sum_{k=0}^{\infty} P(\text{ultimate extinction} | X_1 = k) P(X_1 = k)\end{aligned}$$

i.e., the population becomes extinct iff families started by each of these k members become extinct. It is assumed that the behaviour of each of the families is independent of each other. Therefore,

$$P(\text{ultimate extinction} \mid X_1 = k) = q^k$$

$$\Rightarrow q = \sum_{k=0}^{\infty} q^k p_k = \psi(q)$$

i.e., q satisfies equation (5.6).

Theorem 5.5: Whatever be the value of $E(X_1) = \mu$,

$$\lim_{n \rightarrow \infty} P(X_n = 0) = q$$

where $q = q_n = \lim_{n \rightarrow \infty} P(\text{extinction occurs at or before the } n^{\text{th}} \text{ generation})$

and $\lim_{n \rightarrow \infty} P(X_n = k) = 0 \quad \forall \quad k > 0$

Theorem 5.6: For $r, n = 0, 1, 2, \dots$ $E(X_{n+r} \mid X_n) = X_n \mu^r$.

Proof: The result can be proved with the help of mathematical induction.

For $r = 1$ and all values of n , we have

$$\begin{aligned} E(X_{n+1} \mid X_n) &= E\left(\sum_{i=1}^{X_n} \xi_i \mid X_n\right) \\ &= \sum_{i=1}^{X_n} E(\xi_i) = \mu X_n \end{aligned}$$

Let the result holds for $r = m$, i.e.

$$E(X_{n+m} \mid X_n) = X_n \mu^m$$

For $r = m+1$, we have

$$\begin{aligned}
E(X_{n+m+1} | X_n) &= E(E(X_{n+m+1} | X_{n+k}, X_{n+k-1}, \dots, X_{n+1}) | X_n) \\
&= E(E(X_{n+m+1} | X_{n+k}) | X_n) \\
&= E(\mu X_{n+k} | X_n) \\
&= \mu X_n \mu^m \\
&= X_n \mu^{m+1}
\end{aligned}$$

Thus the result holds for all positive integers $r = 0, 1, 2, \dots$ when it is true for $r = 1$. Hence the theorem.

5.3 Examples

- (1) In the example from genetics, it can be safely assumed that the probability law governing the population of mutant genes is Poisson with mean λ . If $\lambda = 1$, then $\psi(s) = e^{(s-1)}$. Then $q = 1$.

The assumption of a Poisson law can be justified on the basis that a large number of zygotes (fertilized eggs) are subject to an independent Bernoulli law for survival and since mutation of a gene is a rare event, so Poisson distribution can be taken as an approximation to the Binomial distribution. In this case,

$$p_k = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2$$

Now, suppose that $\lambda \sim \text{Gamma}\left(\alpha, \frac{q}{p}\right)$, i.e.

$$f(\lambda) = \begin{cases} \frac{\left(\frac{q}{p}\right)^\alpha \lambda^{\alpha-1} e^{-\left(\frac{q}{p}\right)\lambda}}{\alpha!}, & \lambda \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad 0 < q, p, q+p < 1$$

Then,
$$P(\xi_r = k) = \int_0^\infty P(\xi_r = k | \lambda) f(\lambda) d\lambda$$

Now,

$$\begin{aligned}
\psi(s) &= \sum_{k=0}^{\infty} P(\xi_r = k) s^k \\
&= \sum_{k=0}^{\infty} \left(\int_0^{\infty} P(\xi_r = k | \lambda) f(\lambda) d\lambda \right) s^k \\
&= \sum_{k=0}^{\infty} \left(\int_0^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \frac{\left(\frac{q}{p}\right)^{\alpha} \lambda^{\alpha-1}}{\alpha!} e^{-\left(\frac{q}{p}\right)\lambda} d\lambda \right) s^k \\
&= \int_0^{\infty} e^{-\left(\frac{q}{p}+1-s\right)\lambda} \frac{\left(\frac{q}{p}\right)^{\alpha} \lambda^{\alpha-1}}{\alpha!} d\lambda \\
&= \left(\frac{\frac{q}{p}}{\frac{q}{p}+1-\alpha} \right)^{\alpha} \\
&= \left(\frac{q}{1-ps} \right)^{\alpha}
\end{aligned}$$

which is the p.g.f. of a negative binomial distribution.

- (2) Consider a population where any individual of a generation can give rise to at most two offspring.

We want to verify the above theorem for this population.

If $p_k = P(\text{an object in a generation generates } k \text{ similar objects}), k = 0, 1, 2$

then $\psi(s) = \sum_k p_k s^k = p_0 + p_1 s + p_2 s^2$

Probability of extinction is unity if $\mu \leq 1$ and if $\mu > 1$, then it is the positive root less than unity, of the equation $s = \psi(s)$.

Consider $p_0 = \frac{2}{3}, p_1 = \frac{1}{6}, p_2 = \frac{1}{6}$. Then $\mu = \psi'(1) = p_1 + 2p_2 = \frac{1}{2} < 1$

Now, $s = \psi(s) \Rightarrow s = \frac{2}{3} + \frac{s}{6} + \frac{s^2}{6} \Rightarrow s = 1, 4$

So probability of extinction is unity.

Now, let $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$. Then $\mu = p_1 + 2p_2 = \frac{5}{4} > 1$

and $s = \psi(s) \Rightarrow s = \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2} \Rightarrow s = 1, \frac{1}{2}$. Then the root $s = \frac{1}{2} (< 1)$ is the probability of extinction.

(3) Let $p_k = b(1-b)^k$, $k = 0, 1, 2, \dots$, $0 < b < 1$. Then $\mu = \frac{1-b}{b}$ and

$$\psi(s) = \sum_k b(1-b)^k s^k = \frac{b}{1-s(1-b)}.$$

Then $\psi(s) = s \Rightarrow s = 1, \frac{b}{1-b}$

If $\mu \leq 1$, probability of extinction is unity and if $\mu > 1$, then it is the root $\frac{b}{1-b}$ since in this case

$$\frac{b}{1-b} = \frac{1}{\mu} < 1.$$

(4) Let each individual at the end of one unit of time produces either k (where k is a fixed positive integer greater than or equal to 2) or 0 direct descendents with probability p or q respectively, then

$$\psi(s) = q + ps^k \text{ and}$$

$$\psi(s) = s \Rightarrow q + ps^k = s$$

$$\Rightarrow ps^k - s + (1-p) = 0$$

$$\Rightarrow p(s^k - 1) - (s-1) = 0$$

$$\Rightarrow (s-1) \left(p(s^{k-1} + s^{k-2} + \dots + s + 1) - 1 \right) = 0$$

$$\Rightarrow s = 1, \text{ or } ps^{k-1} + ps^{k-2} + \dots + ps + p - 1 = 0 \quad (5.7)$$

Now, $s \in (-1, 1)$ so only those roots of (5.7) are permissible which lie in this interval. Since 1 is not a repeated root of (5.7) so absolute value of other roots will be less than unity and the probability of extinction, hence, is unity.

In particular, consider the case when $k = 3$, $p = q = \frac{1}{2}$. Then $\psi(s) = \frac{1}{2} + \frac{s^3}{2}$

$$s = \psi(s) \Rightarrow 2s = s^3 + 1$$

$$\Rightarrow (s-1)(s^2+s-1) = 0$$

$$\Rightarrow s = 1, \frac{-1 \pm \sqrt{5}}{2}$$

and discarding the negative root, we have $s = 1, \frac{-1 + \sqrt{5}}{2}$. Then the probability of extinction is $\frac{-1 + \sqrt{5}}{2}$.

Problems

1. In the example from genetics, if p_k obeys a Binomial distribution with parameters n and p , i.e.,

$$p_0 = q; \quad p_n = p; \quad p_k = 0 \quad \forall k \neq 0, n.$$

is the probability distribution of $\{p_k\}$. If X_n be the size of the population at the n^{th} generation, then show that $\{X_n, n \geq 1\}$ is a branching process.

2. If, in the above exercise, p_k obeys a Poisson distribution with parameter λ , find the generating functions $\psi(s)$ and $\psi_n(s)$. What is the distribution of X_n ?
3. As a general social tradition, the name of a family goes by the sons. Let p_k is the probability of having k ($k = 0, 1, 2, \dots$) sons. Then find the probability of extinction if the only possible values of k are 0 and 4. Under what conditions is the population sure to die out?
4. For the probability generating function $\psi(s) = 1 - p(1-s)^\beta$, where p and β are constants having values between 0 and 1, prove that

$$\psi_n(s) = 1 - p^{1+\beta+\dots+\beta^{n-1}} (1-s)^{\beta^n}, \quad n = 1, 2, \dots$$

5. If the offspring distribution of a G.W. branching process is geometric $\{q^k p, k = 0, 1, 2, \dots\}$, show that

$$\psi_n(s) = \begin{cases} \frac{p(q^n - p^n) - pqs(q^{n-1} - p^{n-1})}{(q^{n+1} - p^{n+1}) - (q^n - p^n)qs}, & p \neq q \\ \frac{n-(n-1)s}{(n+1)-ns}, & p = q \end{cases}, \quad n = 1, 2, \dots$$

6. From the relation $\psi_n(s) = \psi(\psi_{n-1}(s))$, find $E(X_n)$ and $Var(X_n)$.
7. If, in a branching process, $\mu = E(X_1) < 1$, then show that $E\left(\sum_{n=1}^{\infty} X_n\right) = \frac{\mu}{1-\mu}$.
8. Let $p_0 = \alpha$; $p_1 = 1 - \alpha - \beta$; $p_2 = \beta$ and $p_k = 0 \forall k \neq 0, 1, 2$. Find the conditions under which the population will ultimately be extinct.
9. Let $\psi(s) = p + qs$; $p + q = 1$; $0 < p < 1$. Find the probability of ultimate extinction. With what probability will the population survive at the n^{th} time point ($n = 1, 2, \dots$)? Find the probability generating function of the probability of survival at the n^{th} time-point.
10. In Lotka's model, $p_k = bc^{k-1}$; $k = 1, 2, \dots$; $0 < b, c, b + c < 1$ and $p_0 = \sum_{k=1}^{\infty} p_k$. Find $\psi(s)$, $\psi_n(s)$, μ and σ^2 . Find also the probability of ultimate extinction.