**Chapter 4**

# Continuous Time Markov Chains

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**Key words:** Continuous time stochastic processes, Poisson process, birth process, death process, generalized birth-death process, successive occurrences, inter-arrival time.

**Suggested readings:**

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2. Feller, W.(1968), An introduction to Probability Theory and its Applications, Vol. I ,Wiley, New York, 3rd edition.
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4. Parzen, E.(1962), Introduction to Stochastic Processes, Universal Book Stall.
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## 5.1 Introduction

Consider the following processes:

(i) A radioactive system emits a stream of $\alpha$ - particles, which reach a Geiger counter. A Geiger counter is a defective counter, which gets locked for sometime when a radioactive particle strikes it. While an unlocked counter is capable of recording the arrival of particles, a locked counter does not record the event. Let $X(t)$ be the number of recordings by a Geiger counter during the time-interval $(0,t]$. Suppose that the half-life of the particle (the time taken till the mass of the matter reduces to its half by the process of disintegration) is large as compared to t. Then $X(t)$ is the sum of a large number of i.i.d. Bernoulli events, each having a probability $p$ (say) of being recorded.

(ii) A large pond has a large number of fishes. Let $X(t)$ be the number of fishes caught in the time-interval $(0,t]$. The chance of catching a fish is independent of the number of fishes already caught, when the catching of a fish is a Bernoulli trial with the probability $p$ (say) of catching (success). Further the chances of catching a fish at the next time point are the same irrespective of the time interval since the last success. In other words, there is “no premium for waiting”.

(iii) Recall the queuing system having one server who serves the customer on the basis of “First come. First served” rule. Let $X(t)$ be the length of the queue at any time $t$, when the customers are joining the system at a constant rate and hence the inter-arrival times are i.i.d. random variables.

All the above processes represent a situation when the number of trials is large, and each trial is subject independently to a Bernoulli law. The probability of one occurrence of the event in a very small interval is constant but in the same interval two or more events can occur with a probability, which is of order zero. These all processes are the Poisson processes.

### 4.2 Poisson process

Let $\left(N(t),\ t \in T = [0,\infty)\right)$ be a non-negative counting process with discrete state space $S = \{0,1,2\ldots\}$ where $N(t)$ denotes the occurrence of a random, rare event $E$ in time $T$. Further, let

$$p_n(t) = P(N(t) = n) = P(\text{of } n \text{ occurences of } E \text{ in an interval } (0,t]); n = 0,1,2...$$

$$\sum_{n=0}^{\infty} p_n(t) = 1$$

i.e., $p_n(t)$ is a function of time. Under certain conditions, called as the postulates or assumptions, $N(t) \sim \text{Pois}(\lambda t)$. Then $\left(N(t),\ t \in T = [0,\infty)\right)$ is called a Poisson process.

**Postulates of Poisson process:** There are three basic postulates or assumptions of the Poisson process:

**(i)** **Independence:** $N(t)$ is Markovian, i.e., the occurrences of the event in $(0,t]$ are independent of the number of occurrences of $E$ in an interval prior to $(0,t]$.

**(ii)** **Homogeneity in time:** $p_n(t)$ depends only on the length $t$ of the time interval $(0,t]$ and not on where the interval is situated on the time-axis.

**(iii)** **Regularity (Orderliness):** In an interval $(t, t + h)$ of infinitesimal length $h$, exactly one event can occur with probability $\lambda h + o(h)$ and the probability of more than one events is of order $o(h)$ (called as the zero order), i.e.,

$$p_1(h) = \lambda h + o(h);\ \lim_{h \to 0} \frac{o(h)}{h} = 0$$

and, $$\sum_{k=2}^{\infty} p_k(h) = o(h)$$

Since $$\sum_{n=2}^{\infty} p_n(h) = 1 \Rightarrow p_0(h) + p_1(h) + o(h) = 1$$

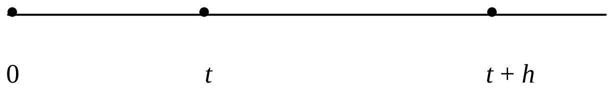
$$\Rightarrow p_0(h) = 1 - \lambda h - o(h) = P(\text{no event in } (t, t+h))$$

Under these postulates, we prove the following theorem:

**Theorem 4.1:** Under the conditions of independence, homogeneity and orderliness, $N(t) \sim \text{Pois}(\lambda t)$, i.e.,

$$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{\lfloor\underline{n}}, \quad n = 0, 1, 2...$$

**Proof:** Consider $p_n(t+h)$ for $n \geq 0$




For $n \geq 1$,

$$p_n(t+h) = P(\, n \text{ events occur in an interval } (0, t+h))$$

$$= P(\, n \text{ events occur in the interval } (0,t))\,P(\, no \text{ events occur in } (t,t+h))$$

$$+ \; P(\, n{-}1 \text{ events occur in the interval } (0,t))\,P(\, one \text{ event occurs in } (t,t+h))$$

$$+ \sum_{k=2}^{n-2} P(\, n-k \text{ events occur in the interval } (0,t))\,P(\, k \text{ events occur in } (t,t+h))$$

$$= p_n(t)(1-\lambda h) + p_{n-1}(t)\lambda h + o(h)$$

$$\Rightarrow \frac{p_n(t+h)-p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(h)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{p_n(t+h)-p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \lim_{h\to 0}\frac{o(h)}{h}$$

or, $\quad p_n'(t) + \lambda p_n(t) = \lambda p_{n-1}(t); \quad n \geq 1 \qquad (4.1)$

For $n = 0$,

$$p_0(t+h) = p_0(t)\,p_0(h) = p_0(t)(1-\lambda h + o(h))$$

$$\Rightarrow \lim_{h\to 0}\frac{p_0(t+h)-p_0(t)}{h} = -\lambda p_0(t)$$

$$\Rightarrow \frac{p_0'(t)}{p_0(t)} = -\lambda$$

or, $\log p_0(t) = -\lambda t + c$, $c$ is the arbitrary constant of integration

$$\Rightarrow p_0(t) = c_1 e^{-\lambda t}, \quad c_1 = e^c$$

Initially, at $t = 0,\ p_0(0) = 0 \Rightarrow c_1 = 1$

$$p_0(t) = e^{-\lambda t} \tag{4.2}$$

Putting $n$ = 1 in (4.1), we have

$$p_1'(t) + \lambda p_1(t) = \lambda p_0(t) = \lambda e^{-\lambda t}$$

Integrating factor I.F. of this differential equation is is $e^{\lambda t}$

$$\Rightarrow \frac{d}{dt}\left(p_1(t)e^{\lambda t}\right) = \lambda$$

$$\Rightarrow p_1(t)e^{\lambda t} = \lambda t + c$$

At $t = 0,\ p_1(0) = 0 \Rightarrow c = 1$

$$\Rightarrow p_1(t) = \lambda t e^{-\lambda t}$$

Let $$p_{n-1}(t) = e^{-\lambda t}\frac{(\lambda t)^{n-1}}{\underline{|n-1}} \tag{4.3}$$

Then from (4.1)

$$p_n'(t) + \lambda p_n(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{\underline{|n-1}}$$

Multiplying both sides by $e^{\lambda t}$, we have

$$e^{\lambda t}(p_n'(t) + \lambda p_n(t)) = \lambda\frac{(\lambda t)^{n-1}}{\underline{|n-1}}$$

Integrating both sides with respect to *t*

$$e^{\lambda t} p_n(t) = \frac{(\lambda t)^n}{\lfloor n} + c$$

$$p_n(0) = 0 \Rightarrow c = 0$$

$$\Rightarrow p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{\lfloor n}, \quad n = 1, 2, 3, ...$$

Thus, by mathematical induction, we have shown that

$$N(t) \sim \text{Pois}(\lambda t), \quad n = 0, 1, 2...$$

**Probability generating function and the Characteristic functions of a Poisson process:**

An alternate and more elegant technique of obtaining the result is the **generating function technique**.

Define the probability generating function

$$P(s,t) = \sum_{n=0}^{\infty} p_n(t) s^n$$

$$= \sum_{n=0}^{\infty} P(N(t)=n) s^n = E(s^{N(t)})$$

so,

$$P(s,0) = \sum_{n=0}^{\infty} p_n(0) s^n = p_0(0) + s p_1(0) + s^2 p_2(0) + \cdots$$

$$= p_0(0)$$

Differentiating $P(s, t)$ partially w.r.t. *t*, we have

$$\frac{\partial P(s,t)}{\partial t} = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} p_n(t) s^n = \sum_{n=0}^{\infty} p_n'(t) s^n$$

$$= p_0'(t) + \sum_{n=1}^{\infty} p_n'(t) s^n \tag{4.4}$$

From (4.1), we have $\quad p_n'(t) + \lambda p_n(t) = \lambda p_{n-1}(t); \quad n \geq 1$

Multiplying this equation by $s^n$ and summing over all possible values of $n$, we have

$$\sum_{n=1}^{\infty} p_n'(t)s^n + \lambda \sum_{n=1}^{\infty} p_n(t)s^n = \lambda \sum_{n=1}^{\infty} p_{n-1}(t)s^n \tag{4.5}$$

where,

$$\sum_{n=1}^{\infty} p_n'(t)s^n = \frac{\partial}{\partial t}P(s,t) - p_0'(t)$$

$$\sum_{n=1}^{\infty} p_n(t)s^n = P(s,t) - p_0(t) \qquad \text{(from (4.4))}$$

$$\sum_{n=1}^{\infty} p_{n-1}(t)s^n = sP(s,t)$$

so, (4.5) becomes

$$\frac{\partial}{\partial t}P(s,t) - p_0'(t) = -\lambda\left(P(s,t) - p_0(t)\right) + \lambda sP(s,t)$$

$$\Rightarrow \frac{\partial}{\partial t}P(s,t) = \lambda(s-1)P(s,t) \qquad (p_0'(t) = \lambda p_0(t))$$

$$\Rightarrow \quad P(s,t) = ce^{\lambda(s-1)t}$$

As $\quad P(s,0) = 0$

$$\Rightarrow \quad P(s,t) = e^{\lambda(s-1)t}$$

Hence, the p.g.f. of the process is given by

$$P(s,t) = e^{\lambda(s-1)t} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda ts)^n}{\underline{|n}}$$

$$\Rightarrow \; p_n(t) = \text{coefficient of } s^n \text{ in } P(s,t)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{\underline{|n}}, \quad n \geq 0$$

$$\Rightarrow \quad N(t) \sim \text{Pois}(\lambda t), \quad n = 0,1,2...$$

The characteristic function of this process is given by

$$\phi_i(s) = E\left(e^{isX(t)}\right) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda ts)^n e^{isn}}{\underline{|n}}$$

$$= e^{\lambda t(e^{is}-1)}$$

**Deductions:**

(i)

$$E(N(t)) = \lambda t$$

$$\text{Var}(N(t)) = \lambda t$$

i.e., the mean and the variance of $N(t)$ depend on $t$ and as such, the process is evolutionary.

(ii) In an interval of unit length, the mean number of occurrences is λ. This is called the parameter of the process.

(iii) Poisson process is a continuous parameter, discrete state space stochastic process. But $E(N(t))$ is a continuous non-random function of $t$.

(iv) Poisson process has independent and stationary (time-homogeneous) increments.

(v) If $E$ occurred $r$ times up to the initial instant 0 from which $t$ is measured then the initial condition will be

$$p_r(0) = 1, \quad p_n(0) = 0 \;\; \forall \; n \neq r$$

$$\Rightarrow \quad p_n(t) = P(N(t) = n-r)$$

$$= \begin{cases} e^{-\lambda t}\dfrac{(\lambda t)^{n-r}}{\underline{|n-r}}, & n \geq r \\ 0, & n < r \end{cases}$$

(vi) As $t \to \infty$, for a Poisson process $N(t)$

$$P\left(\left|\frac{N(t)}{t}\right| - \lambda \geq \varepsilon\right) \to 0, \text{ where } \varepsilon > 0 \text{ is a preassigned number.}$$

Using Chebyshev's inequality for a random variable X,

$$P(|X - E(X)| \geq a) \leq \frac{\text{Var}(X)}{a^2}; \quad a > 0$$

Put $X = N(t)$ $\quad \Rightarrow P(|N(t) - \lambda t| \geq a) \leq \frac{\lambda t}{a^2}; \quad a > 0$

$$\text{or, } P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \frac{a}{t}\right) \leq \frac{\lambda t}{a^2}$$

Let $\frac{a}{t} = \varepsilon$ $\quad \Rightarrow P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \varepsilon\right) \leq \frac{\lambda}{\varepsilon t}$

$$\Rightarrow \lim_{t\to\infty} P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \varepsilon\right) = 0$$

i.e., for large $t$, $\frac{N(t)}{t}$ can be taken as an estimate of $\lambda$.

## 4.3 Properties of Poisson process

(i) **Additive property:** Sum of two independent Poisson processes is again a Poisson process.

**Proof:** Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$ respectively and $N(t) = N_1(t) + N_2(t)$. Then

$$P(N(t) = n) = \sum_{r=0}^{n} P(N_1(t) = r,\ N_2(t) = n-r)$$

$$= \sum_{r=0}^{n} P(N_1(t) = r)P(N_2(t) = n-r) \qquad \text{due to independence}$$

$$= \sum_{r=0}^{n} e^{-\lambda_1 t}\frac{(\lambda_1 t)^r}{\lfloor r}e^{-\lambda_2 t}\frac{(\lambda_2 t)^{n-r}}{\lfloor n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{(\lambda_1 t)^r}{\lfloor r} \frac{(\lambda_2 t)^{n-r}}{\lfloor n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} t^n \left( \frac{\lambda_2{}^n}{\lfloor n} + \frac{\lambda_1 \lambda_2{}^{n-1}}{\lfloor 1 \lfloor n-1} + \frac{\lambda_1{}^2 \lambda_2{}^{n-2}}{\lfloor 2 \lfloor n-2} + \cdots \right)$$

$$= e^{-(\lambda_1+\lambda_2)t} t^n \left( \frac{(\lambda_1+\lambda_2)^n}{\lfloor n} \right), \quad n \geq 0$$

$$= e^{-(\lambda_1+\lambda_2)t} \left( \frac{((\lambda_1+\lambda_2)t)^n}{\lfloor n} \right), \quad n \geq 0$$

$$\Rightarrow \; N(t) \sim \text{Pois}((\lambda_1 + \lambda_2)t)$$

Alternatively, let the p.g.f. of $N_i(t),\ i = 1, 2$ be $E(s^{N_i(t)}) = e^{\lambda_i (s-1)t}$ .

The p.g.f. of $N(t)$ is

$$E(s^{N(t)}) = E(s^{N_1(t)+N_2(t)})$$

$$= E(s^{N_1(t)}) E(s^{N_2(t)})$$

$$= e^{\lambda_1 (s-1)t} e^{\lambda_2 (s-1)t}$$

$$= e^{(\lambda_1+\lambda_2)(s-1)t}$$

$$\Rightarrow \; N(t) \sim \text{Pois}((\lambda_1 + \lambda_2)t)$$

The c.f. of $N(t)$ is

$$\phi_i(s) = E\left(e^{isN(t)}\right) = E\left(e^{is(N_1(t)+N_2(t))}\right)$$

$$= E\left(e^{is(N_1(t)}\right) E\left(e^{is(N_2(t)}\right)$$

$$= e^{(\lambda_1+\lambda_2)t(e^{is}-1)}$$

(ii) **Difference of two independent Poisson processes.**

**Proof:** Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$ respectively and $N(t) = N_1(t) - N_2(t)$ . Then

$$P(N(t)=n) \;=\; \sum_{r=0}^{\infty} P(N_1(t)=n+r,\; N_2(t)=r)$$

$$= \sum_{r=0}^{n} e^{-\lambda_1 t}\frac{(\lambda_1 t)^{n+r}}{\lfloor n+r} e^{-\lambda_2 t}\frac{(\lambda_2 t)^{r}}{\lfloor r}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} \sum_{r=0}^{\infty}\frac{\left(t\sqrt{\lambda_1\lambda_2}\right)^{n+2r}}{\lfloor r\;\lfloor n+r}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_{[n]}\left(t\sqrt{\lambda_1\lambda_2}\right)$$

where, $I_{[n]}(x)=\sum_{r=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{n+2r}}{\lfloor r\;\lfloor n+r}$ is modified Bessel function of order $n$ ($\geq$ -1). $N$ ($t$) is not a Poisson process.

Alternatively, the p.g.f. of $N$ ($t$) is

$$E(s^{N(t)}) \;=\; E(s^{N_1(t)-N_2(t)})$$

$$= E(s^{N_1(t)})E(s^{-N_2(t)})$$

$$= E(s^{N_1(t)})E\left(\frac{1}{s}\right)^{-N_2(t)}$$

$$= e^{(\lambda_1+\lambda_2)t}e^{\left(\lambda_1 s+\frac{\lambda_2}{s}\right)t}$$

Then $p_n(t)$ is the coefficient of $s^n$ in the expansion of $E(s^{N(t)})$

$$E(N(t)) \;=\; (\lambda_1-\lambda_2)t$$

$$E(N^2(t)) \;=\; (\lambda_1+\lambda_2)t+(\lambda_1-\lambda_2)^2t^2$$

$$\Rightarrow\; \mathrm{Var}(N(t)) \;=\; (\lambda_1+\lambda_2)t$$

The c.f. of $N$ ($t$) is

$$\phi_i(s) \;=\; E\left(e^{isN(t)}\right) \;=\; E\left(e^{is(N_1(t)-N_2(t))}\right)$$

$$= E\left(e^{is(N_1(t)}\right)E\left(e^{-is(N_2(t)}\right)$$

$$= e^{t(\lambda_1 e^{is}-\lambda_2 e^{-is})-t(\lambda_1-\lambda_2)}$$

(iii) **Decomposition of a Poisson process:** A random selection from a Poisson process

Let $N(t)$, the number of occurrences of an event $E$ in an interval of length $t$ is a Poisson process with parameter $\lambda$. Further, let each occurrence of $E$ has a constant probability p of being recorded, and that recording of an occurrence is independent of that of another occurrence and of $N(t)$.

If $M(t)$ is the number of occurrences recorded in an interval of length $t$, then $M(t)$ is also a Poisson process with parameter $\lambda p$.

**Proof:**

$$P(M(t)=n) = \sum_{r=0}^{\infty} P(E \text{ occurs } (n+r) \text{ times by epoch } t \text{ and exactly } n \text{ out of } n+r \text{ occurrences are recorded})$$

$$= \sum_{r=0}^{\infty} P(N(t)=n+r)P(n \text{ events are recorded})$$

$$= \sum_{r=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^{n+r}}{\lfloor n+r}\frac{\lfloor n+r}{\lfloor n \, \lfloor r} p^n q^r$$

$$= e^{-\lambda t}(\lambda t)^n \frac{p^n}{\lfloor n}\sum_{r=0}^{\infty}\frac{(\lambda q t)^r}{\lfloor r}$$

$$= e^{-\lambda t}\frac{(\lambda t p)^n}{\lfloor n} e^{\lambda q t}$$

$$= e^{-\lambda t(1-q)}\frac{(\lambda t p)^n}{\lfloor n} = e^{-\lambda t p}\frac{(\lambda t p)^n}{\lfloor n}$$

$$\Rightarrow M(t) \sim \text{Pois}(\lambda p t)$$

The c.f. of $M(t)$ is

$$\phi_i(s) = E\left(e^{isM(t)}\right) = e^{\lambda p t(e^{is}-1)}$$

**Corollary:**

1. If $M_1(t)$ is the number of events not being recorded, then $M_1(t)$ is a Poisson process with parameter $\lambda(1-p) = \lambda q$. For example, a Geiger counter records radioactive disintegrations according to a Poisson law. Also the disintegrations, which have not been recorded, follow a Poisson law.

2. If a Poisson process can be broken up into $r$ independent streams with probabilities $p_1, p_2, \ldots p_r$ ; $\sum_{i=1}^{r} p_i = 1$, then, these $r$ independent streams are Poisson processes with parameters $\lambda p_1, \lambda p_2, \ldots \lambda p_r$ respectively.

(iv) **Poisson process and Binomial distribution:** If $N(t)$ is a Poisson process then for $s < t$

$$P(N(s) = k \mid N(t) = n) = \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$$

**Proof:**

$$P(N(s) = k \mid N(t) = n) = \frac{P(N(s) = k, N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(N(s) = k, N(t-s) = n-k)}{P(N(t) = n)}$$

$$= \frac{P(N(s) = k)P(N(t-s) = n-k)}{P(N(t) = n)}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{\lfloor k} \; \dfrac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-k}}{\lfloor n-k}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{\lfloor n}}$$

$$= \frac{\lfloor n}{\lfloor k \; \lfloor n-k} \frac{s^k (t-s)^{n-k}}{t^n}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k}$$

(v) If $\{N(t), t \geq 0\}$ is a Poisson process then the (auto) correlation coefficient between $N(t)$ and $N(t + s)$ is $\sqrt{\dfrac{t}{t+s}}$.

**Proof:** Let $\lambda$ be the parameter of the process, then

$$E(N(T)) = \lambda T; \quad \mathrm{Var}(N(T)) = \lambda T$$

$$E(N^2(T)) = \lambda T + (\lambda T)^2; \qquad T = t, t+s.$$

$$E(N(t)N(t+s)) = E(N(t)(N(t+s) - N(t) + N(t)))$$

$$= E(N^2(T)) + E(N(t)(N(t+s) - N(t)))$$

$$= E(N^2(T)) + E(N(t))E((N(t+s) - N(t)))$$

as $N(t)$ and $N(t+s)$ are independent.

$$= \lambda t + \lambda^2 t^2 + \lambda^2 ts$$

$$\Rightarrow \mathrm{Cov}(N(t), N(t+s)) = \lambda t + \lambda^2 t^2 + \lambda^2 ts - \lambda t \lambda (t+s)$$

$$= \lambda t$$

$$\Rightarrow \mathrm{Autocor}(N(t), N(t+s)) = \frac{\lambda t}{\sqrt{\lambda t \lambda (t+s)}} = \sqrt{\frac{t}{t+s}}$$

In general, $$\rho(N(t), N(t')) = \sqrt{\frac{\min(t,t')}{\max(t,t')}}$$

**Example**

(1) **M/G/1 queue:** Recall the queuing system where customers join the system hoping for some service. There is a server who serves one customer (if any present) only at time points 0,1,2… The number of customers $Y_n$ in the time interval $(n, n+1)$, are i.i.d. random variables .The service station has a capacity of at most $c$ customers including the one being served and further arrivals are not entertained by the service station (lost customers). Further the service times of successive arrivals are assumed to be independent random variables with a common distribution, say, G. and they are independent of further arrivals. Then $\{X_n, n \geq 1\}$, the number of customers at time point n is a Markov chain with state space $S = \{0,1,2\ldots c\}$. We have

$$X_{n+1} = \begin{cases} Y_n, & if\ X_n = 0 \text{ and } 0 \le Y_n \le c-1 \\ X_{n-1} + Y_n, & if\ 1 \le X_n \le c \text{ and } 0 \le Y_n \le c+1-X_n \\ c, & \text{otherwise} \end{cases}$$

If $Y_n$ is a Poisson process, then

$$P(Y_n = j) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{\lfloor j} dG(x); \quad j = 0, 1, ..$$

and the transition probabilities of the Markov chain $\{X_n, n \ge 0)$ are

$$p_{ij} = \begin{cases} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{\lfloor j} dG(x); & i = 0, j \ge 0 \\ \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{\lfloor j-i+1} dG(x); & j \ge i-1, i \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

## 4.4 Poisson distribution and related distributions

Taking a cue from the above example, we can now identify some distributions, which are closely associated with the Poisson process.

**Inter-arrival time:** Let $\{N(t), t \ge 0\}$ is a Poisson process with parameter $\lambda$. Let $X$ be the interval between two successive occurrences of the event $E$ for which $N(t)$ is the counting process. Then $X$, known as the inter-arrival time is a random variable following an exponential distribution.

We state and prove the following result.

**Theorem 4.2:** The interval between two successive occurrences of a Poisson process $\{N(t), t \ge 0\}$ with parameter $\lambda$ has a negative exponential distribution with mean $\frac{1}{\lambda}$.

**Proof:** Let $X$ be the interval between two successive occurrences of $\{N(t), t \geq 0\}$. Let $F_X(x) = P(X \leq x)$ be the c.d.f. of X.

Let $E_i$ and $E_{i+1}$ be the two successive occurrences of the event $E$ for which $N(t)$ is the counting process occurring at time epochs $t_i$ and $t_{i+1}$ respectively. Then,

$$P(X > x) = P(E_{i+1} \text{ did not occur in } (t_i, t_{i+x}) \mid E_i \text{ occurred at instant } t_i)$$

$$= P(E_{i+1} \text{ did not occur in } (t_i, t_{i+x}) \mid N(t_i) = i)$$

$$= P(N(x) = 0 \mid N(t_i) = i)$$

$$= p_0(x) = e^{-\lambda x}, \quad x > 0$$

Since $i$ is arbitrary, so for the interval $X$ between any two successive occurrences

$$F_X(x) = P(X \leq x) = 1 - P(X > x)$$

$$= 1 - e^{-\lambda x}; \quad x > 0$$

$$\Rightarrow \quad f(x) = \frac{dF_X(x)}{dx} = \lambda e^{-\lambda x}; \quad x > 0$$

$$\Rightarrow \quad X \sim \exp(\lambda)$$

The next result is an extension of this result.

**Theorem 4.3:** The intervals between successive occurrences of a Poisson process are i.i.d. exponential variables with common mean $\frac{1}{\lambda}$.

**Proof:** We prove the result by mathematical induction. Let the successive occurrence points of the Poisson process are $0 < t_1 < t_2 < \dots$. In the earlier theorem, we have proved that the inter-arrival time between two successive occurrences is an exponential variable with mean $\frac{1}{\lambda}$. For three successive occurrences, if $X_i = t_{i+1} - t_i$, $i = 1,2$ are the inter-arrival times, then

$$P(X_1 \le x_1, X_2 > x_2) = P(E_{i+2} \text{ didnot occur in } (t_{i+1}, t_{i+1+x_2})|E_{i+1} \text{ occured at } t_{i+1})$$

$$= \int_0^{x_1} P(t_{i+2} > x_1 + x_2 \mid X_1 = x_1) f(x) dx$$

$$P(t_2 > x_1+x_2| X_1 = x_1) = P(N(t_2) = 0) = e^{-\lambda x_2}$$

$$P(X_1 \le x_1, X_2 > x_2) = e^{-\lambda x_2} \int_0^{x_1} \lambda e^{-\lambda x} dx$$

$$= e^{-\lambda x_2} (1 - e^{-\lambda x_1})$$

Let the result holds for $k$ inter-arrival times $X_1, X_2, ..., X_k$. Then

$$P(X_1 \le x_1, X_2 \le x_2, \cdots X_k \le x_k, X_{k+1} > x_{k+1})$$

$$= \lambda^k \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_k} P(W_{k+1} > \sum_{i=1}^{k+1} x_i \mid X_1 = x_1, X_2 = x_2, \cdots X_k = x_k) e^{-\lambda \sum_{i=1}^{k} x_i} dx_1 \cdots dx_k$$

where

$$P(W_{k+1} > \sum_{i=1}^{k+1} x_i | X_1 = x_1, X_2 = x_2, \cdots X_k = x_k) = P\left( N\left( \sum_{i=1}^{k+1} x_i \right) - N\left( \sum_{i=1}^{k} x_i \right) = 0 \right)$$

$$= P(N(x_{k+1}) = 0)$$

$$\Rightarrow P(X_1 \le x_1, X_2 \le x_2, \cdots X_k \le x_k, X_{k+1} > x_{k+1}) = e^{-\lambda x_{k+1}} (1 - e^{-\lambda x_1}) \cdots (1 - e^{-\lambda x_k})$$

$\Rightarrow$ $X_1, X_2, ..., X_k$ are i.i.d. random variables.

The converse of this theorem is equally true, which along with this theorem gives a characterization to the Poisson process.

**Theorem 4.4:** If the intervals between successive occurrences of an event $E$ are independently distributed exponentially with common mean $\frac{1}{\lambda}$, then the event $E$ has Poisson process as its counting process.

**Proof:** Let $\{Z_i\,,\ \ i \geq 0\}$ be a sequence of i.i.d. negative exponential variables with common mean $\frac{1}{\lambda}$ , where,

$$Z_i \;=\; \text{interval between } (n\text{-}1)^{th} \text{ and } n^{th} \text{ occurrences of the event } E.$$

Define $W_n = Z_1 + Z_2 + ... + Z_n$ as the waiting time up to the $n^{th}$ occurrence, i.e., the time from origin to the $n^{th}$ subsequent occurrence.

Then, $W_n \sim \text{Gamma}(\lambda, n)$ with p.d.f.

$$g_{W_n}(x) \;=\; \frac{\lambda^n x^{n-1} e^{-\lambda x}}{n!}, \quad x > 0$$

and c.d.f. $$F_{W_n}(x) \;=\; P(W_n \leq t) \;=\; \int_0^t g(x)dx$$

Obviously, $\{N(t) < n\} \;=\; \{W_n \;=\; Z_1 + Z_2 + ... + Z_n \;>\; t\}$, i.e. the two c.d.f.'s $F_{N(t)}$ and $F_{W_n}$ satisfy the relation

$$F_{W_n}(t) \;=\; P(W_n \leq t) \;=\; 1 - P(W_n > t)$$

$$= \; 1 \;-\; P(N(t) \;<\; n)$$

$$= \; 1 \;-\; P(N(t) \;\leq\; n\text{–}1)$$

$$= \; 1 \;-\; F_{N(t)}(n\text{–}1)$$

$$\Rightarrow \quad F_{N(t)}(n-1) \;=\; 1 \;-\; F_{W_n}(t)$$

$$= \; \int_0^t \frac{\lambda^n}{n!} x^{n-1} e^{-\lambda x} dx$$

$$= \; \int_t^\infty \frac{\lambda^n}{n!} x^{n-1} e^{-\lambda x} dx$$

$$= \; \int_{\lambda t}^\infty \frac{1}{n!} y^{n-1} e^{-y} dy \qquad (\text{put } \lambda x = y)$$

$$= \sum_{j=0}^{n} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \qquad \text{(integration by parts)}$$

$$\Rightarrow p_n(t) = P(N(t) = n) = F_{N(t)}(n) - F_{N(t)}(n-1)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$\Rightarrow N(t) \sim \text{Pois}(\lambda t); \; t \geq 0$$

It may be noted that Poisson process has independent exponentially distributed inter-arrival times and Gamma distributed waiting times.

The next result explains the purely random nature of a Poisson process.

**Theorem 4.5:** If a Poisson process $N(t)$ has occurred only once by the time-point $t$, then the distribution of the time interval $\gamma$ in $[0,T]$, in which it occurred, is uniform in $[0,T]$, i.e.,

$$P(t < \gamma \leq t + dt \mid N(T) = 1) = \frac{dT}{T}; \; 0 < t < T$$

**Proof:** We have

$$P(t < \gamma \leq t + dt \mid N(T) = 1) = \lambda e^{-\lambda t} dt$$

$$P(N(T) = 1) = \lambda T e^{-\lambda T}$$

and $P(N(T) = 1 \mid \gamma = t) = e^{-\lambda(T-t)}$ is the probability that there was no occurrence of $N(t)$ in the time interval $(t, T]$. Hence,

$$P(t < \gamma \leq t + dt \mid N(T) = 1) = \frac{P(t < \gamma \leq t+dt, N(T)=1)}{P(N(T) = 1)}$$

$$= \frac{P(t < \gamma \leq t+dt) P(N(T) = 1)}{P(N(T) = 1)}$$

$$= \frac{\lambda e^{-\lambda t} dt . e^{-\lambda(T-t)}}{e^{-\lambda T} \lambda T}$$

$$= \frac{dt}{T}$$

The result can be interpreted as follows:

If a Poisson process $N(t)$ has occurred only once by the time-point $t$, this is equally likely to happen anywhere in $[0,T]$. This is why the Poisson process is purely random.

We state some more results, which further emphasize the random nature of Poisson process.

(i) For a Poisson process with parameter $\lambda$, the time interval up to the first occurrence also follows an exponential distribution with mean $\frac{1}{\lambda}$, i.e., if $X_0$ is the time up to the first occurrence, then

$$\begin{aligned} P(X_0 > x) &= P(N(x) = 0) \\ &= p_0(x) = e^{-\lambda x}, \quad x > 0 \end{aligned}$$

i.e., $P(X > x) = e^{-\lambda x}$ is independent of $i$ and $t$.

(ii) Suppose that the interval X is measured from an arbitrary point $t_{i+r}$ $(r > 0)$ in the interval $(t_i, t_{i+1})$ and not the point $t_i$ of the occurrence of $E_i$. Let $Y = t_{i+1} - (t_i + r)$. $Y$ is called random modification of $X$ or the residual time of $X$. Then, if $X$ is exponentially distributed, so is its random modification $Y$ with the same mean. In other words, there is no premium for "waiting".

(iii) Suppose that $A$ and $B$ are two independent series of Poisson events with parameters $\lambda_1$ and $\lambda_2$ respectively. Define a random variable $N$ as the number of occurrences of $A$ between two successive occurrences of $B$. Then

$$N \sim \text{Geo}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right).$$

Let X be the random variable denoting the interval between two successive occurrences of $B$. Then $f_X(x) = \lambda_2 e^{-\lambda_2 x},\ x > 0$. Hence,

$P(A$ occurs $k$ times in an arbitrary interval between two successive occurrences of $B) = P(N = k)$

$$= \int_0^\infty e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{\lfloor k} f(t)\,dt$$

$$= \int_0^\infty e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{\lfloor k} \lambda_2 e^{-\lambda_2 t}\,dt$$

$$= \lambda_2 \frac{\lambda_1^k}{\lfloor k} \int_0^\infty t^k e^{-(\lambda_1+\lambda_2)t}\,dt$$

$$= \frac{\lambda_2 \lambda_1^k}{(\lambda_1+\lambda_2)^{k+1}}$$

$$= \frac{\lambda_2}{(\lambda_1+\lambda_2)} \left( \frac{\lambda_1}{\lambda_1+\lambda_2} \right)^k ; \quad k = 0, 1, 2, \ldots$$

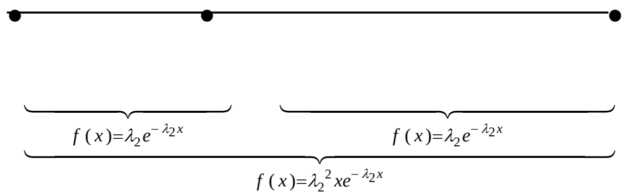
(iv) The above property can be generalized to define what we call as a Poisson count process.

**Poisson count process:** Let $E$ and $E'$ be two random sequences of events occurring at instants $(t_1, t_2, \ldots)$ and $(t_1', t_2', \ldots)$ respectively. The number, $N_n$, of occurrences of $E'$ in an interval $(t_{n-1}, t_n)$ is known as the count process of $E'$ in $E$.

If $E$ is a Poisson process, then the count process is called the Poisson count process.

If, along with $E$, $E'$ is also a Poisson process then the count process $N_n$ has a geometric distribution. $N_n$ (n = 1,2…) are i.i.d. geometric variates.

(v) Suppose that $A$ and $B$ are two independent series of Poisson events with parameters $\lambda_1$ and $\lambda_2$ respectively. Define a random variable $N$ as the number of occurrences of $A$ between two successive occurrences of $B$. the interval between two consecutive occurrences of $B$ is the sum of two independent exponential variates and has the joint density $f(x) = \lambda_2^2 x e^{-\lambda_2 x}$




$$\therefore\ P(k \text{ occurrences of } A \text{ between every second occurrence of } B) = \int_0^\infty e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{\underline{|k}}\lambda_2^2 t e^{-\lambda_2 t}dt$$

$$= \frac{\lambda_1{}^k\lambda_2^2}{\underline{|k}}\int_0^\infty e^{-(\lambda_1+\lambda_2)t}t^{k+1}dt$$

$$= \frac{\lambda_1{}^k\lambda_2^2}{\underline{|k}}\frac{\underline{|k+2}}{(\lambda_1+\lambda_2)^{k+2}}$$

$$= \binom{k+1}{1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k;\ k = 0, 1, 2\cdots$$

i.e., the distribution is negative binomial (convolution of exponential distribution).

## 4.5 Generalizations of Poisson process

In the classical Poisson process, it is assumed that the conditional probabilities are constant, i.e., the probability of $k$ events in the interval $[t, t+h]$ given occurrence of $n$ events by time-point $t$ is given by

$$p_k(h) = P(N(h) = k \mid N(t) = n) = \begin{cases} \lambda h + o(h), & k = 1 \\ o(h), & k \geq 2 \\ 1-\lambda h + o(h), & k = 0 \end{cases}$$

i.e., $p_k(h)$ is independent of $n$ as well as $t$. This process can be generalized by considering $\lambda$ no more a constant but a function of $n$ or $t$ or both. The generalized process is again Markovian in nature.

This generalized process has excellent interpretations in terms of birth-death processes. Consider a population of organisms, which reproduce to create similar organisms. The population is dynamic as there are additions in terms of births and deletions in terms of deaths. Let $n$ be the size of the population at instant $t$. Depending upon the nature of additions and deletions in the population, various types of processes can be defined.

**4.5.1 Pure birth process:** Let $\lambda$ is a function of *n*, the size of the population at instant *t*. Then

$$
\begin{aligned}
p(k,h\,|\,n,t) &= P(N(h)=k\,|\,N(t)=n)\\
&= \begin{cases} \lambda_n h + o(h), & k=1\\ o(h), & k\geq 2\\ 1-\lambda_n h + o(h), & k=0 \end{cases}
\end{aligned}
\tag{4.6}
$$

Then,

$$p_n(t+h) = p_n(t)(1-\lambda_n h) + p_{n-1}(t)\lambda_{n-1}h + o(h), \quad n\geq 1$$

$$p_0(t+h) = p_0(t)(1-\lambda_0 h) + o(h)$$

$$\Rightarrow \quad p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t), \quad n\geq 1 \tag{4.7}$$

$$p_0'(t) = -\lambda_0 p_0(t) \tag{4.8}$$

This is a pure birth process (only births are there and no deaths as *k* is a non-negative integer). For specified initial conditions, an explicit expression for $p_n(t)$ can be obtained.

Depending upon form of $\lambda_n$, different processes can be obtained.

(i) **Yule-Furry process:** Let $\lambda_n = n\lambda$. Then (4.7) and (4.8) can be written as

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t), \quad n\geq 1$$

$$p_0'(t) = 0$$

Let the initial conditions be $p_1(0) = 1$; $p_i(0) = 0\ \forall\ i\neq 1$, i.e., the process starts with only one member at time $t = 0$.

Using principle of mathematical induction, we now, obtain an expression for $p_n(t)$.

For $n = 1$,

$$p_1'(t) = -\lambda p_1(t) \ \Rightarrow\ p_1(t) = c_1 e^{-\lambda t};$$

$c_1$ is the constant of integration.

At $t = 0,\ p_1(0) = 1 \Rightarrow c_1 = 1$

$$\Rightarrow\ p_1(t) = e^{-\lambda t}$$

For $n = 2$,

$$p_2'(t) = -2\lambda p_2(t) + \lambda p_1(t)$$

$$\Rightarrow\ p_2'(t) + 2\lambda p_2(t) = \lambda e^{-\lambda t}$$

Integrating factor for this equation is $e^{2\lambda t}$

$$\Rightarrow\ e^{2\lambda t} p_2(t) = \int_t \lambda e^{2\lambda t} e^{-\lambda t} dt = c_2 + e^{\lambda t}$$

Since $p_2(0) = 0 \Rightarrow c_2 = -1$

$$\Rightarrow\ p_2(t) = e^{-\lambda t}(1 - e^{-\lambda t})$$

Let $p_{n-1}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-2}$

Now,

$$p_n'(t) + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t)$$

$$= (n-1)\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-2}$$

Multiplying both sides by $e^{n\lambda t}$, we have

$$e^{n\lambda t} p_n(t) = \lambda(n-1)\int_t e^{n\lambda t} e^{-\lambda t}(1 - e^{-\lambda t})^{n-2}\, dt = c_n + e^{(n-1)\lambda t}(1 - e^{-\lambda t})^{n-1}$$

Since $p_n(0) = 0 \Rightarrow c_n = 0$

$$\Rightarrow\ p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1};\ n \geq 1$$

and $p_0(t) = 0$

$\Rightarrow\ \{p_n(t), n \geq 1\}$ has geometric distribution with parameter $e^{-\lambda t}$ and p.g.f.

$$P(s,t) = \sum_{n=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{n-1} s^n$$

$$= \frac{se^{-\lambda t}}{1-s(1-e^{-\lambda t})}$$

$$E(N(t)) = P'(s,t)|_{s=1} = e^{\lambda t}$$

$$\text{Var}(N(t)) = e^{\lambda t}(e^{\lambda t}-1)$$

**4.5.2 Birth and death process:** Now, along with additions in the population, we consider deletions also, i.e., along with births, deaths are also possible. Define

$$q(k,h\,|\,n,t) = P(\text{number of deaths in } (t,t+h) = k \mid N(t) = n)$$

$$= \begin{cases} \mu_n h + o(h), & k = 1 \\ o(h), & k \geq 2 \\ 1-\mu_n h + o(h), & k = 0 \end{cases} \tag{4.9}$$

For $n \geq k$, $\mu_0 = 0$

(4.6) and (4.9) together constitute a birth and death process. The probability of more that one birth or more than one death is *o* (*h*). We wish to obtain

$$p_n(t) = P(N(t) = n)$$

To obtain the differential-difference equation for $p_n(i)$, we consider the time interval

$$(0,t+h) = (0,t) + [t,t+h)$$

Since, births and deaths, both are possible in the population, so the event $\{N(t+h) = n,\ n \geq 1\}$ can occur in the following mutually exclusive ways:

$$E_{ij} = n-i+j \text{ individuals at time-point } t,\ i \text{ births and } j \text{ deaths in } (t,t+h)$$
$$i,\ j = 0,1,2...$$

It is easy to see that $P(E_{ij}, i+j > 2) = o(h)$. Therefore,

$$p_n(t+h) = P(E_{00}) + P(E_{10}) + P(E_{01}) + P(E_{11})$$

where,

$$P(E_{00}) = P(\text{ no birth and no death in } (t, t+h) \mid N(t) = n)$$

$$= p_n(t)(1-\lambda_n h + o(h))(1-\mu_n h + o(h))$$

$$= p_n(t)(1-(\lambda_n + \mu_n)h + o(h))$$

$$P(E_{10}) = p_{n-1}(t)(\lambda_{n-1} h + o(h))(1-\mu_{n-1} h + o(h))$$

$$= p_{n-1}(t)(\lambda_{n-1} h + o(h))$$

$$P(E_{01}) = p_{n+1}(t)(1-\lambda_{n+1} h + o(h))(\mu_{n+1} h + o(h))$$

$$= p_{n+1}(t)(\mu_{n+1} h + o(h))$$

$$P(E_{11}) = p_n(t)(\lambda_n h + o(h))(\mu_n h + o(h))$$

$$= p_n(t)(o(h))$$

$$= o(h)$$

So for $n \geq 1$

$$p_n(t+h) = p_n(t)(1-(\lambda_n + \mu_n)h) + p_{n-1}(t)\lambda_{n-1}h + p_{n+1}(t)\mu_{n+1}h + o(h)$$

$$\Rightarrow p_n'(t) = -(\lambda_n + \mu_n) p_n(t) + \lambda_{n-1} p_{n-1}(t) + \mu_{n+1} p_{n+1}(t) \qquad (4.10)$$

For $n = 0$

$$p_0(t+h) = p_0(t)(1-\lambda_0 h + o(h))(1-\mu_0 h + o(h)) + p_1(t)(1-\lambda_1 h + o(h))(\mu_1 h + o(h)$$

$$= p_0(t) - \lambda_0 h p_0(t) + \mu_1 h p_1(t) + o(h)$$

$$\Rightarrow p_0'(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t) \qquad (4.11)$$

Initially at $t = 0$, if $i$ organisms are there in the population, then

$$p_n(0) = 0, \quad n \neq i$$

$$p_i(0) = 1$$

(4.10) and (4.11) represent the differential-difference equations of a birth and death process.

We make the following assertion:

For arbitrary $\lambda_n \geq 0$, $\mu_n \geq 0$, there **always** exists a solution $p_n(t)(\geq 0)$ such that $\sum_n p_n(t) \leq 1$. If $\lambda_n$ and $\mu_n$ are bounded, the solution is unique and satisfies $\sum_n p_n(t) = 1$.

**4.5.3 Births and death rates:** Depending upon the values of $\lambda_n$ and $\mu_n$, various types of birth and death processes can be defined.

(i) **Immigration:** When $\lambda_n = \lambda$, i.e., $\lambda_n$ is independent of population size *n*, then the increase in the population can be regarded as due to an external source. The process is, then, known as an immigration process.

(ii) **Emigration:** When $\mu_n = \mu$, i.e., $\mu_n$ is independent of population size *n*, then the decrease in the population can be regarded as due to elimination of some elements present in the population. The process is, then, known as an emigration process.

(iii) **Linear birth process:** When $\lambda_n = n\lambda$, then $\lambda_n h = n\lambda h$, is the conditional probability of one birth in an interval of length *h*, given that *n* organisms are present at the beginning of the interval. $\lambda_1 = \lambda$ is the birth rate in a unit interval per organism. $\lambda_0 = 0$.

(iv) **Linear death process:** When $\mu_n = n\mu$, then the process is known as a linear death process.

When the specific values of both $\lambda_n$ and $\mu_n$ are considered simultaneously, we get the following processes:

(i) **Immigration-emigration process:** When $\lambda_n = \lambda$ and $\mu_n = \mu$, the process is known as immigration-emigration process. This is an M/M/1 queue.

(ii) **Linear growth process:** If for a birth and death process

$$P(\text{an element of the population gives birth to a new member in a small interval of length } h)$$

$$= \lambda h + o(h)$$

$$\Rightarrow P(\text{ one birth in interval } (t, t+h) \mid N(t) = n) = n\lambda h + o(h)$$

and

$$P(\text{ an element of the population dies in a small interval of length } h) = \mu h + o(h)$$

$$\Rightarrow P(\text{ one death in an interval } (t, t+h) \mid N(t) = n) = n\mu h + o(h)$$

i.e., if for a birth and death process, $\lambda_n = n\lambda$ and $\mu_n = n\mu$ $(n \geq 1)$; $\lambda_0 = \mu_0 = 0$, then the process is a linear growth process.

This process, which is evolutionary in nature, has extensive applications in various fields, particularly, in queuing theory. Now we shall analyze this process.

The differential-difference equations for this process are

$$p_n'(t) = -n(\lambda + \mu) p_n(t) + (n-1)\lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t), \quad n \geq 1 \tag{4.12}$$

and
$$p_0'(t) = \mu p_1(t) \tag{4.13}$$

(a) **Generating function:** Let the p.g.f. of $\{p_n(t)\}$ be

$$P(s,t) = \sum_{n=0}^{\infty} p_n(t) s^n$$

Then,
$$\frac{\partial}{\partial s} P(s,t) = \sum_{n=1}^{\infty} n p_n(t) s^{n-1}$$

and
$$\frac{\partial}{\partial t} P(s,t) = \sum_{n=1}^{\infty} p_n'(t) s^n$$

Multiplying (4.12) by $s^n$, summing over $n = 1,2,3\ldots$ and then adding (4.13) to the result, we have

$$p_0'(t) + \sum_{n=1}^{\infty} p_n'(t) s^n = -(\lambda + \mu) \sum_{n=1}^{\infty} p_n(t) s^n + \lambda \sum_{n=1}^{\infty} (n-1) p_{n-1}(t) s^n + \mu \sum_{n=1}^{\infty} (n+1) p_{n+1}(t) + \mu p_1(t)$$

$$\Rightarrow \quad \frac{\partial P}{\partial t} \;=\; -(\lambda+\mu)s\frac{\partial P}{\partial s}+\lambda s^2\frac{\partial P}{\partial s}+\mu\frac{\partial P}{\partial s}$$

$$= \;(\mu-(\lambda+\mu)s+\lambda s^2)\frac{\partial P}{\partial s}$$

Under the initial condition $N(0) = i$, the solution of this partial differential-difference equation is given by

$$P(s,t) \;=\; \left(\frac{\mu(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}{\lambda(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}\right)^i \tag{4.14}$$

Expanding $P(s,t)$ as a power series in $n$, we get $p_n(t)$.

(b) **Mean population size:** Differentiating $P(s,t)$ w.r.t. $s$ partially at $s = 1$, we get the mean population size $M(t)$ as

$$M(t) \;=\; \frac{\partial P(s,t)}{\partial t} \;=\; ie^{-(\lambda-\mu)t}$$

$$\text{As } t\to\infty,\ M(t) \;=\; \begin{cases} 0, & \text{if } \lambda<\mu \\ \infty, & \text{if } \lambda>\mu \\ i, & \text{if } \lambda=\mu \end{cases}$$

Since this method involves differentiation of a not-so-easy p.g.f., so obtaining $M(t)$ may be a bit involved exercise. Alternatively, $M(t)$ can be obtained from (4.12) and (4.13) directly.

Now,
$$M(t) \;=\; E(N(t)) \;=\; \sum_{n=1}^{\infty} np_n(t)$$

Multiplying both sides of (4.12) by $n$ and adding over different values of $n$, we have

$$\sum_{n=1}^{\infty} np_n{}'(t) \;=\; -(\lambda+\mu)\sum_{n=1}^{\infty} n^2p_n(t)+\lambda\sum_{n=1}^{\infty} n(n-1)p_{n-1}(t)+\mu\sum_{n=1}^{\infty} n(n+1)p_{n+1}(t) \tag{4.15}$$

where,

$$\sum_{n=1}^{\infty} n(n-1)p_{n-1}(t) \;=\; \sum_{n=1}^{\infty}(n-1)^2p_{n-1}(t)+\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)$$

$$= \;M_2(t)+M(t)$$

where,
$$M_2(t) \;=\; E(N^2(t)) \;=\; \sum_{n=1}^{\infty} n^2p_n(t)$$

and,

$$\sum_{n=1}^{\infty} n(n+1)\,p_{n+1}(t) \;=\; \sum_{n=1}^{\infty}(n+1)^2\,p_{n+1}(t)-\sum_{n=1}^{\infty}(n+1)\,p_{n+1}(t)$$

$$=\;(M_2(t)-p_1(t))-(M(t)-p_1(t))$$

$$=\;M_2(t)-M(t)$$

and, $$\sum_{n=1}^{\infty} np_n{}'(t) \;=\; M'(t)$$

Therefore, from (4.15), we get

$$M'(t) \;=\; -(\lambda+\mu)M_2(t)+\lambda\left(M_2(t)+M(t)\right)+\mu\left(M_2(t)-M(t)\right)$$

$$=\quad(\lambda-\mu)M(t)$$

$\Rightarrow\; M(t) \;=\; ce^{(\lambda-\mu)t}$, $c$ being the constant of integration.

Initially, $M(0) \;=\; i \;\Rightarrow\; c \;=\; i \Rightarrow\; M(t) \;=\; ie^{(\lambda-\mu)t}$

Again, from (4.12)

$$\sum_{n=1}^{\infty} n^2 p_n{}'(t) \;=\; -(\lambda+\mu)\sum_{n=1}^{\infty} n^3 p_n(t)+\lambda\sum_{n=1}^{\infty} n^2(n-1)\,p_{n-1}(t)+\mu\sum_{n=1}^{\infty} n^2(n+1)\,p_{n+1}(t)$$

$$\Rightarrow\; M_2{}'(t) \;=\; -(\lambda+\mu)M_3(t)+\lambda\sum_{n=1}^{\infty}\Big((n-1)^3+(n-1)^2+n(n-1)\Big)p_{n-1}(t)$$

$$+\mu\sum_{n=1}^{\infty}\Big((n+1)^3-(n+1)^2-n(n+1)\Big)p_{n+1}(t)$$

$$\Rightarrow\; M_2{}'(t) \;=\; 2(\lambda-\mu)M_2(t)+(\lambda+\mu)M(t)$$

or, $$M_2{}'(t)-2(\lambda-\mu)M_2(t) \;=\; (\lambda+\mu)ie^{-(\lambda-\mu)t}$$

$$\Rightarrow\quad M_2(t)e^{-2(\lambda-\mu)t} \;=\; \frac{i(\lambda+\mu)}{\lambda-\mu}e^{-(\lambda-\mu)t}+c$$

Initially, $M_2(0) \;=\; i^2$

$$\Rightarrow \quad c = i^2 - \frac{i(\lambda+\mu)}{\lambda-\mu}$$

$$\Rightarrow \quad M_2(t)e^{-2(\lambda-\mu)t} = \frac{i(\lambda+\mu)}{\lambda-\mu}e^{-(\lambda-\mu)t} + i^2 - \frac{i(\lambda+\mu)}{\lambda-\mu}$$

$$\mathrm{Var}(N(t)) = \frac{i(\lambda+\mu)}{\lambda-\mu}e^{(\lambda-\mu)t}\left(e^{(\lambda-\mu)t}-1\right), \text{ if } \lambda \neq \mu$$

If $\lambda = \mu$, then

$$M_2'(t) = (\lambda+\mu)M(t) = i.2\lambda$$

$$\Rightarrow M_2(t) = i.2\lambda t + c$$

At $t=0$, $\quad M_2(0) = i^2$

$$\Rightarrow M_2(t) = i.2\lambda t + i^2$$

$$\text{and } \mathrm{Var}(N(t)) = 2i\lambda t$$

(c) **Probability of extinction:** Since $\lambda_0 = 0$, so 0 is an absorbing state, i.e. once the population reaches 0, it remains thereafter and the population becomes extinct.

Without any loss of generality, let $N(0) = 1$. Then (4.14) becomes

$$P(s,t) = \frac{\mu(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}{\lambda(1-s)-(\mu-\lambda s)e^{-(\lambda-\mu)t}}$$

$$= \frac{\mu(1-e^{-(\lambda-\mu)t})-s(\mu-\lambda e^{-(\lambda-\mu)t})}{(\lambda-\mu e^{-(\lambda-\mu)t})-\lambda s(1-e^{-(\lambda-\mu)t})}$$

$$= \frac{a-bs}{c-ds} = \frac{\frac{a}{c}\left(1-\frac{bs}{a}\right)}{1-\frac{ds}{c}}$$

where,

$$a = \mu(1-e^{-(\lambda-\mu)t})$$

$$b = \mu-\lambda e^{-(\lambda-\mu)t}$$

$$c = \lambda-\mu e^{-(\lambda-\mu)t}$$

$$d = 1-e^{-(\lambda-\mu)t}$$

So, $\quad \dfrac{a}{c} = \dfrac{\mu(1-e^{-(\lambda-\mu)t})}{\lambda-\mu e^{-(\lambda-\mu)t}} = P(N(t)=0) = p_0(t)$

$$P(\text{ the population will eventually become extinct}) = \lim_{t\to\infty} p_0(t)$$

$$= \lim_{t\to\infty} \frac{\mu(1-e^{-(\lambda-\mu)t})}{\lambda-\mu e^{-(\lambda-\mu)t}} = \begin{cases} \dfrac{\mu}{\lambda} < 1, \text{ if } \lambda > \mu \\ 1, \quad\quad \text{if } \lambda > \mu \end{cases}$$

and $\quad \lim_{t\to\infty} p_n(t) = 0$ for $n \neq 0$ if $\lambda < \mu$.

The physical interpretation of the probability of extinction is that if the birth rate is less than the death rate in a population, the population will ultimately become extinct with probability 1. If birth rate is more than the death rate, then the population becomes extinct with probability less than unity.

(iii) **Linear growth with immigration:** For linear growth, $\lambda_0 = 0$ and once the population reaches 0, it is bound to remain there itself and 0 becomes an absorbing state. However, if we assume that along with births, additions in the population are possible through immigrations also, i.e., some organisms from some other populations may also join the population under consideration, then $\lambda_n = n\lambda + a \ (a > 0)$ and $\mu_n = n\mu \ (n \geq 1)$; $\lambda_0 = \text{a}, \mu_0 = 0$ and state 0 is no longer an absorbing state. As soon as the birth rate reaches 0, some other organisms join the system and the population will never become extinct (reflecting barriers). This process is the process of linear growth with immigration.

(iv) **Immigration-death process:** If $\lambda_n = \lambda$ and $\mu_n = n\mu \ (n \geq 0)$, then the birth rate is constant and death rate is a linear function of *n*, then the process is known as immigration-death process. This is the self-service model of the queuing theory ($M / M / \infty$).

(v) **Pure death process:** In this case, $\lambda_n = 0 \ \forall \ n$, i.e., no new births other than those present at the beginning of the process, are possible and

$$P(\text{ of a death in } (t, t+h)) \;=\; \mu h + o(h)$$

$$\text{so, } P(\text{ of a death in } (t, t+h) \mid N(t) = n) \;=\; n\mu h + o(h)$$

This process is called a pure death process.

Now, for this process $\lambda_n = 0$ and $\mu_n = n\mu$ $(n \geq 0)$

$$\Rightarrow \quad p_n(t+h) \;=\; p_n(t)(1 - n\mu h) + p_{n+1}(t)((n+1)\mu h) + o(h)$$

$$\Rightarrow \quad p_n'(t) \;=\; -n\mu p_n(t) + (n+1)\mu p_{n+1}(t), \quad n \geq 1 \tag{4.16}$$

and,

$$p_0(t+h) \;=\; p_0(t) + p_1(t)(\mu h) + o(h)$$

$$\Rightarrow \quad p_0'(t) \;=\; \mu p_1(t)$$

(4.17)

(4.16) and (4.17) are the differential-difference equations of a pure death process.

To obtain an expression for $p_n(t)$, we assume that initially $i$ individuals were present when the process began.

For $n = i$,

$$p_i'(t) \;=\; -i\mu p_i(t)$$

$$\Rightarrow \quad \frac{p_i'(t)}{p_i(t)} \;=\; -i\mu$$

$$\text{or, } \frac{d}{dt}(\ln p_i(t)) \;=\; -i\mu$$

$$\Rightarrow \quad p_i(t) \;=\; ce^{-i\mu t}, \quad c \text{ being the constant of integration.}$$

Initially, $\quad p_i(0) \;=\; 1 \;\Rightarrow\; c \;=\; 1 \;\Rightarrow\; p_i(t) \;=\; e^{-i\mu t}$

For $n = i$-1,

$$p_{i-1}'(t) \;=\; -(i-1)\mu p_{i-1}(t) + i\mu p_i(t),$$

$$\Rightarrow \quad p_{i-1}'(t) + (i-1)\mu p_{i-1}(t) \;=\; i\mu p_i(t)$$

**Integrating factor for the equation is** $e^{i(\mu-1)t}$

$$\Rightarrow \frac{d}{dt}(e^{(i-1)\mu t} p_{i-1}(t)) = i\mu e^{-\mu t}$$

$$\text{or,} \quad e^{(i-1)\mu t} p_{i-1}(t) = \int i\mu e^{-\mu t} dt = -ie^{-\mu t} + c$$

At $t = 0, \ c = -i$

$$\Rightarrow p_{i-1}(t) = i(1-e^{-\mu t})\, e^{-(i-1)\mu t} = \binom{i}{1}(1-e^{-\mu t})\, e^{-(i-1)\mu t}$$

Proceeding in the similar manner, we have

$$p_n(t) = \binom{i}{n}(1-e^{-\mu t})^{i-n}\, e^{-n\mu t}; \quad n = 0,1,2...,i$$

Now, we proceed to obtain mean and variance of a pure death process.

Multiplying (4.16) by *n* and summation over all possible values of *n*, we have

$$\sum_{n=1}^{\infty} np_n'(t) = -\mu \sum_{n=1}^{\infty} n^2 p_n(t) + \mu \sum_{n=1}^{\infty} n(n+1)\, p_{n+1}(t)$$

$$= -\mu M_2(t) + \mu\left(M_2(t) - M(t)\right)$$

where,

$$M_2(t) = E(N^2(t)) = \sum_{n=1}^{\infty} n^2 p_n(t)$$

$$\text{and, } M(t) = E(N(t)) = \sum_{n=1}^{\infty} np_n(t)$$

$$\Rightarrow \quad M'(t) = -\mu M(t)$$

$$\Rightarrow \quad M(t) = ce^{-\mu t}$$

Initially, $M(0) = i \ \Rightarrow \ c = i \Rightarrow \ M(t) = ie^{-\mu t}$.

Again, from (4.16)

$$\sum_{n=1}^{\infty} n^2 p_n{}'(t) = -\mu \sum_{n=1}^{\infty} n^3 p_n(t) + \mu \sum_{n=1}^{\infty} n^2 (n+1)\, p_{n+1}(t)$$

$$= -\mu M_3(t) + \mu\Big((n+1)^3 - (n+1)^2 - n(n+1)\Big) p_{n+1}(t)$$

$$= -\mu M_3(t) + \mu\Big(M_3(t) - p_1(t) - M_2(t) + p_1(t) M_2(t) + M(t)\Big)$$

where, $M_3(t) = \sum_{n=1}^{\infty} n^3 p_n(t) \quad \Rightarrow \quad M_2(t) = -2\mu M_2(t) + M(t)$

or, $\quad M_2(t) + 2\mu M_2(t) = i\mu e^{-\mu t}$

$\Rightarrow \quad e^{2\mu t} M_2(t) = ie^{\mu t} + c$, where $c$ is the constant of integration

Now, $M_2(0) = i^2 \Rightarrow c = i^2 - i$

$$\Rightarrow \quad M_2(t) = i\mu e^{-\mu t} + (i^2 - i\mu) e^{-2\mu t}$$

$$\Rightarrow \quad \mathrm{Var}(N(t)) = ie^{-\mu t}(1 - e^{-\mu t})$$

## Problems

1. An investment agent sells short-term investment policies which are, in general, either for one-year or for two-years. He is able to sell policies according to a Poisson process with rate 100 per month when with probability 0.6, he sells one-year policies and with probability 0.4, he sells two-years policies. If he receives a commission 10% for one-year policies and a commission 12% for two-years policies, find his expected gain as a percentage of business done by him. What is the variance of his income?

2. A radioactive source emits particles at a rate 5 per minute in accordance with a Poisson process. Each emitted particle has a probability 0.75 of being recorded. Find the probability that in a 10-minute interval exactly 20 recordings will be there. What is the probability of recording atlas 20 emissions during the same time-period?

3. If $N_1$ ($t$) and $N_2$ ($t$) are two independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$ respectively, then show that

$$P(N_1(t) = k \mid N_1(t) + N_2(t) = n) = \binom{n}{k} p^k q^{n-k}$$

where $$p = \frac{\lambda_1}{\lambda_1+\lambda_2}; \quad q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

4. If, for an interval of very small length $h$, one success can occur with probability $\lambda h$ or no success with probability 1-$\lambda h$, then the limiting distribution when the number of such (non-overlapping) intervals $n$ tends to infinity is a Poisson process.

**5.** Suppose that customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute. Find the probability that the interval between two successive arrivals is

(a) more than one minute (b) less than 4 minutes, and (c) between 1 and 3 minutes.

6. In a nuclear physics experiment, radioactive particles from two different sources have been directed to strike at a screen. Arrivals are according to independent Poisson processes with mean rate λ per minute from the first source and μ per minute from the second source. Then show that the interval between any two successive arrivals has a negative exponential distribution with mean $\frac{1}{\lambda+\mu}$ per minute.

7. For a linear growth process, find the probability of ultimate extinction When the process starts with $i$ individuals at time 0.