

Chapter 3
Markov Chains-II
Babita Goyal

Key words: communicative states, class, class-property, closed set, absorbing states, transient state, persistent state, null state, irreducible chain, ergodicity, stationary distribution, stable Markov chain, gambler's ruin problem.

Suggested readings:

1. Medhi, J. (1996), Stochastic Processes, New Age International (P) Ltd.
2. Feller, W.(1968), An introduction to Probability Theory and its Applications, Vol. I ,Wiley, New York, 3rd edition.
3. Karlin, S. and Taylor, H.M.(1975), A first course in Stochastic Processes, Academic Press.
4. Parzen, E.(1962), Introduction to Stochastic Processes, Universal Book Stall.
5. Ross, S.M.(1983), Stochastic Processes, John Wiley.

3.1 Introduction

In the earlier chapter, we defined a Markov chain and studied various aspects of the Markov chains and the probability distribution associated with these chains. We, now, try to have some deep insight into those aspects of Markov chains and look into the limiting behaviour of these chains. The reason for this is simple as, in general, one is interested in the long term stability of any dynamic system. We will also study some special Markov chains.

3.2 Classification of states and chains

As we have seen that a state can be an absorbing state, i.e., once the system enters that system, it would remain there forever, or from a state transition of the system to another state is possible. The states can further be classified according to their properties and depending upon the classification of states, classification of chains is possible.

- (i) **Communicative states:** If two states i and j are such that $p_{ji}^{(m)} > 0$ for some $m \geq 0$ and $p_{ij}^{(n)} > 0$ for some $n \geq 0$, i.e., the two states can be reached from one another in any number of steps, then i and j are called communicative states ($i \leftrightarrow j$).

The relation is transitive, i.e., if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.

From Chapman-Kolmogorov equation

$$p_{ik}^{(m+n)} = \sum_r p_{ir}^{(m)} p_{rk}^{(n)}$$

So, $p_{ik}^{(m+n)} \geq p_{ij}^{(m)} p_{jk}^{(n)}$ for any intermediate state j .

Also, the relation is symmetric, i.e., $i \leftrightarrow j \Leftrightarrow j \leftrightarrow i$.

- (ii) **Class and class property:** A subset of the state space is called a class if every state within it can communicate with every other state in it and no state outside it can communicate with every state in it.

A property of a state is called a class property if its possession by a state of a class implies its possession by every other state of the class.

- (iii) **Closed set:** If C is a set of states such that no state outside C can be reached from any state within C , then C is said to be a closed set.

i.e., if $j \in C$ and $k \notin C$, then $p_{jk} = 0$; $p_{jk}^{(2)} = 0$; and $p_{jk}^{(n)} = 0 \quad \forall n$.

Alternatively, C is a closed set iff $\sum_{j \in C} p_{ij} = 1 \quad \forall i \in C$. Then the sub matrix $P_1 = (p_{ij}; i, j \in C)$ and the transition probability matrix P can be expressed in the canonical form as

$$P = \begin{pmatrix} P_1 & 0 \\ R_1 & Q \end{pmatrix}$$

- (iv) **Absorbing state:** If a closed set C contains only one state j , then $p_{jj} = 1, p_{jk} = 0 \quad \forall k \neq j$, the state j is called an absorbing state.

For a finite Markov chain, the set of all states is a closed set.

Irreducible chains: If a Markov chain contains no other closed set than the set of all states, i.e., every state can be reached from any state (irrespective of the number of steps) then the chain is called an irreducible chain.

For an irreducible chain all the states belong to the same class.

If a Markov chain is not irreducible, then it is called reducible or decomposable chain.

For an irreducible (reducible) chain, the corresponding transition matrix is also irreducible (reducible).

Suppose that a Markov system starts with state j . Define

$$f_{jk}^{(n)} = P(\text{The system reaches state } k \text{ from state } j \text{ for the first time at the } n^{\text{th}} \text{ trial})$$

$$p_{jk}^{(n)} = P(\text{The system reaches state } k \text{ from state } j \text{ at the } n^{\text{th}} \text{ trial not necessarily for the first time})$$

Let τ_k be the first passage time to state k , i.e., $\tau_k = \min\{n \geq 1, X_n = k\}$

Then, $\{f_{jk}^{(n)}\}$ is the probability distribution of τ_k given that the system starts with state j .

We have the following theorem.

Theorem 3. 1: (First entrance Theorem) For any states j and k

$$p_{jk}^{(n)} = \sum_{r=0}^n f_{jk}^{(r)} p_{kk}^{(n-r)}; \quad n \geq 1$$

$$\text{where,} \quad p_{kk}^{(0)} = 1; \quad f_{jk}^{(0)} = 0; \quad f_{jk}^{(1)} = p_{jk}.$$

Proof:

$$f_{jk}^{(r)} p_{kk}^{(n-r)} = P(\text{starting from state } j, \text{ the state } k \text{ can be reached for the first time at the } r^{\text{th}} \text{ step and after that state } k \text{ again be reached in } n-r \text{ steps, } r \leq n)$$

Since, r is any intermediate state so,

$$p_{jk}^{(n)} = \sum_{r=0}^n f_{jk}^{(r)} p_{kk}^{(n-r)}; \quad n \geq 1$$

Note:

$$p_{jk}^{(n)} = \sum_{r=1}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)} + f_{jk}^{(n)}; \quad n \geq 1$$

$$\Rightarrow f_{jk}^{(n)} = \sum_{r=1}^{n-1} f_{jk}^{(r)} p_{kk}^{(n-r)} - p_{jk}^{(n)}$$

First passage time distribution:

Define $F_{jk} = P(\text{Starting state } j \text{ the system ever reaches state } k)$

$$\therefore F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

Obviously,

$$\sup_{n \geq 1} p_{jk}^{(n)} \leq F_{jk} \leq \sum_{m \geq 1} p_{jk}^{(m)} \quad \forall n \geq 1$$

Define

$$\mu_{jk} = \text{mean time or the first passage time from state } j \text{ to state } k$$

$$\text{i.e.,} \quad \mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}^{(n)}$$

In particular, when $k = j$, then $\{f_{jj}^{(n)}, n \geq 1\}$ is the distribution of recurrence time of state j , i.e.,

$$\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)} \text{ is the recurrence time of state } j.$$

Depending upon the value of F_{jk} , two cases arise:

Case 1: $F_{jk} = 1$: In this case, starting from state j the system will reach state k with probability 1, i.e.,

$\{f_{jk}^{(n)}, n \geq 1\}$ is a proper probability distribution of first passage time of state k (from state j).

In particular, when $k = j$, then $\{f_{jj}^{(n)}, n \geq 1\}$ is the distribution of recurrence time of state j .

$F_{jj} = 1$ means that return to state j is certain. In this case, $\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$ is the mean recurrence time

of state j

Case 2: $F_{jk} < 1$: In this case, $\{f_{jk}^{(n)}, n \geq 1\}$ is not a proper probability distribution and with non-zero probability (< 1), the system will reach state k from state j .

Then, the states can be categorized as follows:

Persistent (recurrent) state: The state j is said to be persistent if return to state j is certain, i.e., $F_{jj} = 1$.

Transient state: If $F_{jj} < 1$, then the state j is said to be a transient state.

Null state: A persistent state j is said to be a (persistent) null state if $\mu_{jj} = \infty$, i.e., the mean recurrence time is infinite.

If $\mu_{jj} < \infty$, the state is called non-null or positive (persistent).

Periodic state: A state j is said to be a periodic state with period $t (>1)$ if return to the state is possible only at $t, 2t, 3t \dots$ steps where t is the greatest integer with this property, i.e.,

$$p_{jj}^{(n)} = 0 \text{ if } n \text{ is not a multiple of } t, \text{ i.e., } t = G.C.D.\{m: p_{jj}^{(m)} > 0\}.$$

If no such $t > 1$ exists, then state j is said to be non-periodic or aperiodic.

Periodicity is a class property.

Ergodic state: A persistent, non-null and aperiodic state of a Markov chain is called an Ergodic state.

Ergodic chain: A Markov chain with all states Ergodic, is called an Ergodic chain.

3.3 Examples

- (1) For the one-dimensional unrestricted random walk, return to equilibrium can occur only at an even number of steps, i.e.,

$$p_{00}^{2n+1} = 0, \quad n = 0, 1, 2, \dots$$

$$p_{00}^{2n} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!n!} p^n q^n$$

By Stirling's approximation,

$$n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{2\pi}, \text{ so}$$

$$p_{00}^{2n} = \frac{(2n)!}{n!n!} p^n q^n = \frac{4(pq)^n}{\sqrt{n\pi}}$$

Then, $\sum_{n=0}^{\infty} p_{00}^n = \infty \Leftrightarrow p = q = \frac{1}{2}$. This is equivalent to saying that in one-dimensional

unrestricted random walk, the origin is a recurrent state iff $p = q = \frac{1}{2}$.

(2) Let $\{X_n, n \geq 1\}$ be a Markov chain having state space $S = \{1, 2, 3, 4\}$ and t.p.m.

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{matrix}$$

We wish to check whether or not all the states Ergodic.

State1:

$$\begin{aligned} f_{11}^{(1)} &= \frac{1}{3}; \\ f_{11}^{(2)} &= f_{12} f_{21} = \frac{2}{3} \\ \Rightarrow F_{11} &= \frac{1}{3} + \frac{2}{3} = 1 \end{aligned}$$

i.e., state 1 is persistent.

$$\mu_{11} = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3} < \infty$$

$\Rightarrow$ state 1 is non-null.

Since $p_{11} = \frac{1}{3} > 0 \Rightarrow$ state 1 is aperiodic.

Hence 1 is an ergodic state.

State2:

$$\begin{aligned} f_{22}^{(1)} &= 0; \\ f_{22}^{(2)} &= f_{21} f_{12} + f_{23} f_{32} + f_{24} f_{42} = 1 \cdot \frac{2}{3} = \frac{2}{3} \\ f_{22}^{(3)} &= f_{21} f_{11} f_{12} = 1 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \end{aligned}$$

$$f_{22}^{(4)} = 1 \cdot \left(\frac{1}{3}\right)^2 \frac{2}{3}$$

$\vdots$

$$f_{22}^{(n)} = 1 \cdot \left(\frac{1}{3}\right)^{n-2} \frac{2}{3}; \quad n \geq 2$$

$$\Rightarrow F_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-2} \frac{2}{3} = 1$$

$\Rightarrow$ state 2 is persistent.

$$\mu_{22} = \sum_{n=1}^{\infty} n f_{22}^{(n)} = \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-2} \frac{2}{3} = 2 \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^{n-1} = 2 \cdot \frac{5}{4} = \frac{5}{2} < \infty$$

$\Rightarrow$ state 2 is non-null aperiodic.

Hence 2 is an ergodic state.

State 3:

$$f_{33}^{(1)} = \frac{1}{2};$$

$$f_{33}^{(2)} = f_{33}^{(3)} = \dots = 0$$

$$\Rightarrow F_{33} = \frac{1}{2} < 1$$

$\Rightarrow$ state 3 is a transient state.

State 4:

$$f_{44}^{(1)} = \frac{1}{2};$$

$$f_{44}^{(2)} = f_{44}^{(3)} = \dots = 0$$

$$\Rightarrow F_{44} = \frac{1}{2} < 1$$

$\Rightarrow$ state 4 is also a transient state.

(3) Consider a Markov chain with the transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{pmatrix} \end{matrix}$$

Then the chain is irreducible.

State 1:

$$f_{11}^{(4)} = f_{13} f_{32} f_{24} f_{41} = 1 \cdot 1 \cdot 1 \cdot \frac{1}{4} = \frac{1}{4} > 0$$

$$f_{12}^{(2)} = f_{13} f_{32} = 1;$$

$$f_{13} = 1$$

$$f_{14}^{(3)} = f_{13} f_{32} f_{24} = 1 \cdot 1 \cdot 1 = 1$$

$\Rightarrow$ from state 1, every other state can be reached.

State 2:

$$f_{22}^{(2)} = f_{24} f_{42} = \frac{1}{8};$$

$$f_{21}^{(2)} = f_{24} f_{41} = \frac{1}{4};$$

$$f_{23}^{(2)} = f_{24} f_{43} = \frac{1}{8};$$

$$f_{24}^{(2)} = 1$$

$\Rightarrow$ from state 2, every other state can be reached

State 3:

$$f_{31}^{(3)} = f_{32} f_{24} f_{41} = \frac{1}{4};$$

$$f_{32} = 1;$$

$$f_{33}^{(3)} = f_{32} f_{24} f_{43} = \frac{1}{8};$$

$$f_{34}^{(2)} = f_{32} f_{24} = 1$$

$\Rightarrow$ from state 3, every other state can be reached

State 4:

$$\begin{aligned}
f_{41} &= \frac{1}{4}; \\
f_{42} &= \frac{1}{8}; \\
f_{43} &= \frac{1}{8}; \\
f_{44} &= \frac{1}{2}
\end{aligned}$$

$\Rightarrow$ from state 3, every other state can be reached

Thus all the four states of the set communicate with each other. Now,

$$p_{44} = f_{44} = \frac{1}{2} > 0 \Rightarrow \text{state 4 is aperiodic}$$

$$f_{44}^{(2)} = f_{42} f_{24} = \frac{1}{8};$$

$$f_{44}^{(3)} = f_{43} f_{32} f_{24} = \frac{1}{8};$$

$$f_{44}^{(4)} = f_{41} f_{13} f_{32} f_{24} = \frac{1}{4}$$

$$\text{and } f_{44}^{(n)} = 0 \quad \forall \quad n > 4$$

$$\Rightarrow F_{44} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = 1 \Rightarrow 4 \text{ is a persistent state.}$$

$$\mu_{44} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{4} = \frac{17}{8} < \infty$$

Hence state 4 is an ergodic state.

Since ergodicity is a class property and the state space is a class, so all the states of the class are ergodic. Hence the Markov chain is an irreducible chain.

The following theorems provide a mechanism for the classification of states.

Theorem 3.2: State j is persistent iff $\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$.

Proof: Let $P_{jj}(s) = \sum_{n=0}^{\infty} p_{jj}^{(n)} s^n = 1 + \sum_{n=1}^{\infty} p_{jj}^{(n)} s^n, \quad |s| < 1$

and
$$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} f_{jj}^{(n)} s^n, \quad |s| < 1$$

be the generating functions of the sequences $\{p_{jj}^{(n)}\}$ and $\{f_{jj}^{(n)}\}$ respectively.

Also, we know that for $n \geq 1$

$$p_{jj}^{(n)} = \sum_{r=0}^n f_{jj}^{(r)} p_{jj}^{(n-r)}$$

Multiplying both sides by s^n and summing over n , we have

$$\begin{aligned} \sum_{n \geq 1} p_{jj}^{(n)} s^n &= \sum_{n \geq 1} \sum_{r=0}^n f_{jj}^{(r)} p_{jj}^{(n-r)} s^n \\ \Rightarrow P_{jj}(s) - 1 &= P_{jj}(s) F_{jj}(s) \\ \Rightarrow P_{jj}(s) &= \frac{1}{1 - F_{jj}(s)}, \quad |s| < 1 \\ \Rightarrow \lim_{s \rightarrow 1} P_{jj}(s) &= \lim_{s \rightarrow 1} \frac{1}{1 - F_{jj}(s)} = \frac{1}{\lim_{s \rightarrow 1} (1 - F_{jj}(s))} \text{ on using Abel's lemma} \\ \Rightarrow P_{jj}(1) &= \frac{1}{1 - F_{jj}(1)} \end{aligned}$$

If j is persistent, then
$$F_{jj}(1) = F_{jj} = 1 \Rightarrow P_{jj}(1) = \sum p_{jj}^{(n)} = \infty.$$

Conversely, if j is transient, then

$$\begin{aligned} F_{jj}(s) &< 1 \\ \Rightarrow \lim_{s \rightarrow 1} F_{jj}(s) &< 1 \\ \Rightarrow \lim_{s \rightarrow 1} F_{jj}(s) &< \infty \\ \Rightarrow \sum_n p_{jj}^{(n)} &< \infty \end{aligned}$$

Remarks: From the above theorem, we can conclude the following:

1. State j is transient if $\sum_n p_{jj}^{(n)} < \infty \Rightarrow \lim_{n \rightarrow \infty} p_{jj}^{(n)} \rightarrow 0$
2. State space of a finite Markov chain must contain at least one persistent state.

3. If k is a transient state and j is an arbitrary state then $\sum p_{jk}^{(n)}$ is convergent and $\lim_{n \rightarrow \infty} p_{jk}^{(n)} \rightarrow 0$.
4. If a Markov chain having a set of transient states T , starts in a transient state, then with probability 1, it stays at T only a finite number of times after which it enters a recurrent state where it remains forever.

We illustrate the result with the help of the following example:

Consider the Markov chain with t.p.m.

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Then,

$$P^2 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}; \quad P_3 = P$$

In general,

$$P^{2n} = P^2; \quad P^{2n+1} = P \\ \Rightarrow p_{ii}^{(2n)} > 0; \quad p_{ii}^{(2n+1)} = 0 \quad \forall i$$

Hence states are periodic with period 2.

Now, $f_{11} = 0$; $f_{11}^{(2)} = 1 \Rightarrow F_{11} = 1 \Rightarrow$ state 1 is persistent. Similarly states 0 and 1 are persistent.

Also, $\mu_{11} = \sum_n n f_{11}^{(n)} = 2 \Rightarrow$ state 1 is non-null. Similarly states 0 and 1 are also non-null.

Now, we state the basic limit theorem of Markov chains:

Lemma: Let $\{f_n\}$ be a sequence such that $f_n \geq 0$; $\sum f_n = 1$ and t (≥ 1) be the greatest common divisor of those n for which $f_n > 0$.

Let $\{u_n\}$ be another sequence such that $u_0 = 0$; $u_n = \sum_{r=1}^n f_r u_{n-r}$ ($n \geq 1$). Then

$$\lim_{n \rightarrow \infty} u_{nt} = \frac{t}{\mu}$$

$$\text{where, } \mu = \sum_{n=1}^{\infty} n f_n .$$

If $\mu \rightarrow \infty$ then $\lim_{n \rightarrow \infty} u_{nt} = 0$. If N is not divisible by t , then $\lim_{N \rightarrow \infty} u_N = 0$.

Theorem 3.3: If state j is persistent non-null, then as $n \rightarrow \infty$

- (i) $p_{jj}^{(nt)} \rightarrow \frac{t}{\mu_{jj}}$ when state j is periodic with period t ; and
- (ii) $p_{jj}^{(n)} \rightarrow \frac{1}{\mu_{jj}}$ when state j is aperiodic.

In case, state j is persistent null (whether periodic or aperiodic), then

$$p_{jj}^{(n)} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Remarks: (i) If state j is persistent non-null, then $\lim_{n \rightarrow \infty} p_{jj}^{(n)} > 0$.

(ii) If state j is persistent null or transient, then $\lim_{n \rightarrow \infty} p_{jj}^{(n)} \rightarrow 0$

Theorem 3.4: If state k is persistent null, then for every j , $\lim_{n \rightarrow \infty} p_{jk}^{(n)} \rightarrow 0$ and if state k is aperiodic,

$$\text{persistent non-null, then } \lim_{n \rightarrow \infty} p_{jk}^{(n)} \rightarrow \frac{F_{jk}}{\mu_{kk}} .$$

Theorem 3.5: In an irreducible chain, all the states are of same type. They are either all transient, all persistent null or all persistent non-null. All the states are periodic with the same period or aperiodic.

Proof: Since the chain is irreducible, so every state can be reached from every other state. If i and j are any states, then

$$p_{ij}^{(N)} = a > 0 \text{ for some } N \geq 1$$

$$\text{and } p_{ji}^{(M)} = b > 0 \text{ for some } M \geq 1$$

We have

$$p_{jk}^{(n+m)} = p_{jk}^{(m+n)} = \sum_r p_{jr}^{(m)} p_{rk}^{(n)} \geq p_{jr}^{(m)} p_{rk}^{(n)} \quad \forall r$$

$$\Rightarrow p_{ii}^{(n+N+m)} \geq p_{ij}^{(N)} p_{jj}^{(n)} p_{ji}^{(m)} = abp_{jj}^{(n)} \quad (i)$$

$$\text{and } p_{jj}^{(n+N+m)} \geq p_{ij}^{(N)} p_{jj}^{(n)} p_{ji}^{(m)} = abp_{ii}^{(n)} \quad (ii)$$

From (i) and (ii) above, it is obvious that the two series $\sum_n p_{ii}^{(n)}$ and $\sum_n p_{jj}^{(n)}$ converge or diverge together.

Thus the two states i and j are either both persistent or both transient.

Suppose that i is persistent null. Then $p_{ii}^{(n)} \rightarrow 0$ as $n \rightarrow \infty \Rightarrow$ from (a) j is persistent null.

Suppose that i is persistent no-null with period $t \Rightarrow p_{ii}^{(nt)} > 0$.

Now, $p_{ii}^{(N+M)} \geq p_{ij}^{(N)} p_{ji}^{(M)} = ab \geq 0 \Rightarrow N+M$ is a multiple of t .

and as $p_{jj}^{(n+N+M)} \geq abp_{ii}^{(n)} \geq 0 \Rightarrow nt + N + M$ is a multiple of $t \Rightarrow t$ is a period of state j also.

Hence the result.

3.4 Stability of a Markov system: Limiting behaviour

We now state some results associated with the limiting behaviour of a stable Markov system. For that, first of all we describe what we mean by a stationary distribution associated with a Markov chain.

Stationary distributions: Consider a Markov chain with the transition probability distribution $\{p_{jk}\}$. A probability distribution $\{\nu_j\}$ is called stationary or invariant for the given chain if

$$\nu_k = \sum_i \nu_i p_{ik} \text{ such that } \nu_i \geq 0; \sum_i \nu_i = 1$$

In this situation

$$\begin{aligned} \nu_k &= \sum_i \nu_i p_{ik} = \sum_i \left(\sum_j \nu_j p_{ji} \right) p_{ik} = \sum_j \nu_j p_{ji}^{(2)} \\ &= \dots = \sum_j \nu_j p_{jk}^{(n)}; \quad n \geq 1 \end{aligned}$$

This phenomenon can be interpreted as a situation that if for a Markov chain a stationary distribution exists, then after a sufficiently large number of steps, the transition probability distribution $\{p_{jk}^{(n)}\}$ becomes independent of the initial state j and the transition probability matrix $P^{(n)}$ tends to have all the rows identical. In such a situation, the system becomes stable and displays some regularity properties. In other words, such a Markov chain is an **ergodic** chain.

A sufficient condition for a Markov chain to be ergodic or equivalently for the existence of a stationary distribution for a Markov chain is given by the following theorem.

Theorem 3.6: (Ergodicity theorem): For a finite, irreducible, aperiodic chain with the transition probability matrix $P = (p_{ij})$, the limits

$$\nu_k = \lim_{n \rightarrow \infty} p_{jk}^{(n)}; \quad \nu_k \geq 0; \quad \sum_k \nu_k = 1$$

exist and are independent of the initial state j . The limiting probability distribution $\{\nu_k\}$ is identical with the stationary distribution for the given chain, so that

$$\nu_k = \sum_i \nu_i p_{ik}; \quad \sum_k \nu_k = 1.$$

The above theorem makes use of the following result.

Theorem 3.7: If the state j is persistent, then for every state k that can be reached from j , $F_{jk} = 1$.

The converse of the theorem (3.6) is also true.

Theorem 3.8: If a chain is irreducible and aperiodic and if there exists a unique stationary distribution $\{\nu_k\}$ for the chain, then the chain is ergodic and $\nu_k = \frac{1}{\mu_{kk}}$.

Thus ergodicity is a necessary and sufficient condition for a Markov chain to have a stationary distribution or to attain stability in long term.

3.5 Some special Markov chains

We, now consider some special Markov chains, which are used very frequently to explain various phenomena in diverse fields and analyze various aspects of these chains.

(i) The classical ruin problem:

Recall the gambler's problem in which one was interested in finding the probability of gambler's ultimate ruin or his ultimate victory. Then 0 and a are the barriers for the gambler which can be absorbing or reflecting barriers, depending upon transition probabilities associated with states 0 and a . Now, we obtain the expressions for ultimate ruin or 0 ultimate victory of gambler.

When the gambler is playing with initial capital z , $1 \leq z \leq a$, let

$$q_z = P(\text{gambler's ultimate ruin})$$

$$p_z = P(\text{gambler's ultimate victory})$$

i.e., if 0 and a are absorbing barriers then q_z and p_z are the probabilities of absorption at 0 and a respectively.

If the game is to end, then $q_z + p_z = 1$.

The difference equation for q_z is

$$q_z = pq_{z+1} + qq_{z-1}; \quad 1 \leq z \leq a-1 \quad (3.1)$$

Here, p is the probability of winning a game and q is the probability of loosing a game. This is the difference equation for termination of the game with the boundary conditions

$$q_0 = 1; \quad q_z = 0$$

$$q_1 = pq_2 + q \tag{3.2}$$

$$q_{a-1} = qq_{a-2}$$

The restriction for q_0 suggests that with initial capital 0, the gambler cannot start playing and he is already ruined. With initial capital a , he is already the winner. So the question of his ruin doesn't arise. For $z = 1$, the player will be ruined if he loses the first game.

To obtain an expression for q_z , put $q_z = \lambda^z$; $\lambda \neq 0$

Then (3.1) is transformed to

$$\begin{aligned} \lambda^z &= p\lambda^{z+1} + q\lambda^{z-1} \\ \Rightarrow \lambda^{z-1}(p\lambda^2 - \lambda + q) &= 0 \\ \Rightarrow p\lambda^2 - \lambda + q &= 0 \\ \Rightarrow \lambda &= \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm |p-q|}{2p} = 1 \text{ or } \frac{q}{p} \\ \Rightarrow q^z &= 1 \text{ or } \left(\frac{q}{p}\right)^z \end{aligned}$$

Hence a general solution of (3.2) is given by

$$q_z = A + B\left(\frac{q}{p}\right)^z \tag{3.3}$$

where,

$$q_0 = A + B = 1$$

and

$$q_a = A + B\left(\frac{q}{p}\right)^a = 0$$

Solving for A and B , we have

$$B = \frac{-1}{1 - \left(\frac{q}{p}\right)^a} \text{ and } A = \frac{\left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^a}$$

$$\Rightarrow q_z = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{1 - \left(\frac{q}{p}\right)^a} \quad (3.4)$$

If $p = q = \frac{1}{2}$, (3.2) changes to

$$2q_z = q_{z+1} + q_{z-1}$$

Put $q_z = \lambda^z$; $\lambda \neq 0$

$$\Rightarrow \lambda^{z-1}(\lambda^2 - 2\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, 1$$

Hence, a general solution is given by

$$q_z = A + Bz$$

$$q_0 = A;$$

$$q_1 = 0 = A + Ba \Rightarrow B = \frac{-1}{a}$$

$$\Rightarrow q_z = 1 - \frac{z}{a} \quad (3.5)$$

The solutions (3.4) and (3.5) are unique. Thus the result can be put as

$$q_z = P(\text{gamlber's ultimate ruin}) = \begin{cases} \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}, & \text{if } p \neq q; \\ 1 - \frac{z}{a}, & \text{if } p = q \end{cases}$$

$$p_z = P(\text{gamlber's ultimate victory}) = \begin{cases} \frac{\left(\frac{p}{q}\right)^a - \left(\frac{p}{q}\right)^z}{\left(\frac{p}{q}\right)^a - 1}, & \text{if } p \neq q \\ \frac{z}{a}, & \text{if } p = q \end{cases}$$

Last expression has been obtained by replacing p , q , and z by q , p , and $a-z$ respectively.

Again,

$$q_z + p_z = \frac{q^a p^z - q^z p^a}{p^z (q^a - p^a)} + \frac{q^{a-z} p^a - q^a p^{a-z}}{q^{a-z} (p^a - q^a)} = 1$$

Now, define a random variable G as

$$G = \begin{cases} a-z & \text{with probability } p_z \\ -z & \text{with probability } q_z \end{cases}$$

i.e., G is the amount which the gambler wins at the end of the game. Therefore,

$$\begin{aligned} E(G) &= (a-z)p_z - zq_z \\ &= a(1-q_z) - z \text{ is the expected gain.} \end{aligned}$$

$$E(G) = 0 \Leftrightarrow q_z = 1 - \frac{z}{a} \Leftrightarrow p = q = \frac{1}{2}$$

Changing stakes: If we change the unit of the stake to its half, it is equivalent to doubling the initial

capital, i.e., $z \rightarrow 2z$; $a \rightarrow 2a$. The new probability of ruin q_z^* will be given by

$$q_z^* = \frac{\left(\frac{q}{p}\right)^{2a} - \left(\frac{q}{p}\right)^{2z}}{\left(\frac{q}{p}\right)^{2a} - 1} = q^z \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$$

If $q > p$, then $q_z^* > q_z$ or $p_z^* < p_z$.

Thus, if stakes are doubled while the initial capital remains unchanged, then the probability of ruin decreases for the player with $p < \frac{1}{2}$ and increases for the other.

When $a \rightarrow \infty$, then the opponent is infinitely rich so

$$q_z = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1} = \frac{1 - \left(\frac{q}{p}\right)^{z-a}}{1 - \left(\frac{q}{p}\right)^{-a}} \xrightarrow{a \rightarrow \infty} \begin{cases} 1, & \text{if } p \leq q \\ \left(\frac{q}{p}\right)^z & \text{if } p > q \end{cases}$$

is the probability of ruin of the gambler when playing against an infinitely rich adversary.

Expected duration of the game

Let D_z be the expected duration of the game, which is assumed to be finite.

If the first game results in a success, then the game continues as if the initial capital with the gambler had been $z + 1$, and, therefore, the expected duration of the game would be $D_{z+1} + 1$.

Similarly, if the first game results in a failure, then the game continues as if the initial capital with the gambler had been $z - 1$, and, therefore, the expected duration of the game would be $D_{z-1} + 1$.

Hence D_z satisfies the difference equation

$$D_z = p(D_{z+1} + 1) + q(D_{z-1} + 1)$$

$$\text{or, } D_z = pD_{z+1} + qD_{z-1} + 1$$

The boundary conditions are $D_0 = 0$; $D_a = 0$

A complete solution of this equation is a general solution + a particular solution.

To obtain a general solution, put $D_z = \lambda^z$

$\therefore$ the auxiliary equation is

$$\lambda^z = p\lambda^{z+1} + q\lambda^{z-1}$$

$$\Rightarrow \lambda^{z-1}(p\lambda^2 - \lambda + q) = 0$$

$$\Rightarrow p\lambda^2 - \lambda + q = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm |p-q|}{2p} = 1 \text{ or } \frac{q}{p}$$

$$\Rightarrow q^z = 1 \text{ or } \left(\frac{q}{p}\right)^z$$

$\therefore$ for $p \neq q$, G.S. for D_z is

$$D_z = A + B\left(\frac{q}{p}\right)^z$$

For particular solution, put $D_z = \lambda z$

$$\Rightarrow \lambda z = p\lambda(z+1) + q\lambda(z-1)$$

$$= \lambda z + \lambda(p-q) + 1$$

$$\Rightarrow \lambda = \frac{1}{q-p} \Rightarrow D_z = \frac{z}{q-p}$$

So a complete solution for D_z is

$$D_z = A + B\left(\frac{q}{p}\right)^z + \frac{z}{q-p}$$

$$\Rightarrow D_0 = A + B = 0$$

$$\text{and } D_a = A + B\left(\frac{q}{p}\right)^a + \frac{a}{q-p}$$

$$\Rightarrow B = \frac{a}{(q-p)\left(1-\left(\frac{q}{p}\right)^a\right)}; \quad A = \frac{-a}{(q-p)\left(1-\left(\frac{q}{p}\right)^a\right)}$$

$$\therefore D_z = \frac{z}{q-p} - \frac{a}{q-p} \frac{\left(1-\left(\frac{q}{p}\right)^z\right)}{\left(1-\left(\frac{q}{p}\right)^a\right)}; \quad q > p$$

When $p = q = \frac{1}{2}$

In this case, the difference equation becomes

$$D_z = \frac{1}{2}D_{z+1} + \frac{1}{2}D_{z-1} + 1$$

Put $D_z = \lambda z^2$

$$\Rightarrow \lambda z^2 = \frac{1}{2}\lambda(z+1)^2 + \frac{1}{2}\lambda(z-1)^2 + 1$$

$$\Rightarrow \lambda = -1$$

$$\Rightarrow D_z = -z^2 \text{ is a particular solution of the equation.}$$

A general solution is $D_z = A + Bz$, hence a complete solution is

$$D_z = A + Bz - z^2$$

$$D_z = A = 0; D_a = A + Ba - a^2 = 0$$

$$\therefore D_z = az - z^2 = z(a - z)$$

If the two players are playing a fair game with initial capital, say, Rs. 1000 each, it will take, on average, 1,00,000 games before one of them is ruined completely.

When $a \rightarrow \infty$

$$D_z = \frac{z}{q-p} - \frac{a}{q-p} \left(\frac{1 - \left(\frac{q}{p}\right)^z}{1 - \left(\frac{q}{p}\right)^a} \right); \quad q \neq p$$

$$= \begin{cases} \frac{z}{q-p}, & \text{if } q > p \\ \infty, & \text{if } p \geq q \end{cases}$$

Generating function of the duration of the game and for the first passage times

Consider the restricted random walk with absorbing barriers at 0 and a . The initial position of the particle is z ($0 < z < a$). Define

$$u_{z,n} = P(\text{the process terminates at the } n^{\text{th}} \text{ step at barrier } 0)$$

After the first step, the position of the particle is $z + 1$ or $z - 1$, and hence, for $1 < z < a-1$, $n \geq 1$,

$$u_{z,n+1} = pu_{z+1,n} + qu_{z-1,n} \quad (3.6)$$

Also define

$$u_{0,n} = u_{a,n} = 0, \quad n \geq 1$$

$$u_{0,0} = 1$$

$$u_{z,0} = 0, \quad z > 0$$

(3.6) implies

$$\sum_{n=0}^{\infty} u_{z,n+1} s^{n+1} = p \sum_{n=0}^{\infty} u_{z+1,n} s^{n+1} + q \sum_{n=0}^{\infty} u_{z-1,n} s^{n+1}; \quad |s| < 1 \quad (3.7)$$

$$\Rightarrow U_z(s) = psU_{z+1}(s) + qU_{z-1}(s); \quad u_{z,0} = 0$$

where, $U_z(s) = \sum_{n=0}^{\infty} u_{z,n} s^n$ is the generating function of $\{u_{z,n}\}$

$$U_0(s) = 1; \quad U_a(s) = 0$$

(3.7) is a homogenous difference equation. Put $U_z(s) = \lambda^z(s)$. Then (3.7) is transformed

to

$$\lambda^z(s) = ps\lambda^{z+1}(s) + qs\lambda^{z-1}(s)$$

$$\Rightarrow \lambda(s) = \frac{1 \pm \sqrt{1-4pqs^2}}{2ps}$$

$$\Rightarrow \lambda_1(s) = \frac{1 + \sqrt{1-4pqs^2}}{2ps}; \quad \lambda_2(s) = \frac{1 - \sqrt{1-4pqs^2}}{2ps}$$

$$\lambda_1(s) + \lambda_2(s) = \frac{1}{ps}; \quad \lambda_1(s) \cdot \lambda_2(s) = \frac{q}{p}$$

Hence a general solution to (3.7) is given by

$$U_z(s) = A(s)\lambda_1^z(s) + B(s)\lambda_2^z(s)$$

$$U_0(s) = A(s) + B(s) = 1$$

$$U_a(s) = A(s)\lambda_1^a(s) + B(s)\lambda_2^a(s) = 0$$

$$\Rightarrow A(s) = \frac{-\lambda_2^a(s)}{\lambda_1^a(s) - \lambda_2^a(s)}; B(s) = \frac{\lambda_2^a(s)}{\lambda_1^a(s) - \lambda_2^a(s)}$$

$$\therefore U_z(s) = \lambda_1^z(s)\lambda_2^z(s) \left(\frac{\lambda_1^{a-z}(s) - \lambda_2^{a-z}(s)}{\lambda_1^a(s) - \lambda_2^a(s)} \right)$$

$U_z(s)$ is the generating function of the probability of ultimate ruin of the gambler, i.e., absorption at 0. At $s = 1$

$$\lambda_1 = \lambda_1(1) = \frac{1 + \sqrt{1-4pq}}{2p} = 1$$

$$\lambda_2 = \lambda_2(1) = \frac{1 - \sqrt{1-4pq}}{2p} = \frac{q}{p}$$

$$\therefore U_z(1) = \sum_n u_{z,n} = q_z = \frac{\left(\frac{q}{p}\right)^z \left(1 - \left(\frac{q}{p}\right)^{a-z}\right)}{1 - \left(\frac{q}{p}\right)^a} = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$$

If, $v_{z,n} = P(\text{the process terminates at the } n^{\text{th}} \text{ step at barrier } a)$ i.e., the probability of ultimate ruin of the opponent, then

$$\lambda_1'(s) = \frac{1 + \sqrt{1-4pqs^2}}{2qs}; \lambda_2'(s) = \frac{1 - \sqrt{1-4pqs^2}}{2qs}$$

$$\lambda_1(s) + \lambda_2(s) = \frac{1}{ps}; \lambda_1(s) \cdot \lambda_2(s) = \frac{q}{p}$$

and

$$\begin{aligned} V_z(s) &= \left(\frac{p}{q}\right)^{a-z} \left(\frac{\left(\frac{p}{q}\lambda_1(s)\right)^z - \left(\frac{p}{q}\lambda_2(s)\right)^z}{\left(\frac{p}{q}\lambda_1(s)\right)^a - \left(\frac{p}{q}\lambda_2(s)\right)^a} \right) \\ &= \frac{\lambda_1^z(s) - \lambda_2^z(s)}{\lambda_1^a(s) - \lambda_2^a(s)} \end{aligned}$$

When $p = q$, $q_z = z(a-z)$.

When $a \rightarrow \infty$, the solution is unbounded at infinity, unless $A(s) = 0$. In that case, unique boundary is $U_0(s) = 1$ and hence

$$U_z(s) = \lambda_2^z(s)$$

So, if $p \leq q$, $q_z = 1$ and if $q > p$, then $q_z = \frac{q}{p}$

(ii) **Generalizations of independent Bernoulli trials:** Sequence of chain dependent trials

Consider a sequence of Markov dependent trials having two outcomes, a success (S) and a failure (F) having the t.p.m.

$$P = \begin{matrix} & \begin{matrix} X_{n-1} & 0 & 1 \end{matrix} \\ \begin{matrix} X_n \\ 0 \\ 1 \end{matrix} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{matrix} ; 0 < a, b < 1; n \geq 1$$

with the initial distribution is given by

$$P(X_0 = 1) = p_1 = 1 - P(X_1 = 1)$$

Denote

$$p_n = P(X_n = 1); n \geq 1$$

and

$$q_n = 1 - p_n = 1 - P(X_n = 1)$$

$$\therefore p_n = P(X_n = 1) = P(X_n = 1, X_{n-1} = 1) + P(X_n = 1, X_{n-1} = 0)$$

$$= p_{n-1}(1-b) + q_{n-1}a$$

$$= (1-a-b)p_{n-1} + a; n \geq 1$$

The solution of this difference equation yields

$$p_n = \frac{a}{a+b} + \left(p_1 - \frac{a}{a+b}\right)(1-a-b)^n; n \geq 1$$

Also,

$$E(X_n) = P(X_n = 1) = p_n = \frac{a}{a+b} + \left(p_1 - \frac{a}{a+b}\right)(1-a-b)^n$$

$$\text{Var}(X_n) = E(X_n^2) - E^2(X_n) = p_n(1-p_n)$$

Remark: When $1-a = b$, then the t.p.m. becomes

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

i.e., the trials are independent and $p_n = a$. Hence the above Markov process is a generalization of independent Bernoulli trials.

(iii) **Markov-Bernoulli chain:** consider a chain with t.p.m.

$$P = \begin{matrix} & \begin{matrix} X_{n-1} & 0 & 1 \end{matrix} \\ \begin{matrix} X_n \\ 0 \\ 1 \end{matrix} & \begin{pmatrix} 1-(1-c)p & (1-c)p \\ (1-c)(1-p) & (1-c)p+c \end{pmatrix} \end{matrix}; 0 < a < 1; 0 \leq c \leq 1$$

with the initial distribution is given by

$$p_1 = p = P(X_0 = 1) = 1 - P(X_0 = 0)$$

When $c = 0$, the t.p.m. becomes

$$\begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$$

i.e., the trials are independent

When $c = 1$, the t.p.m. becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e., the chain remains in its initial state forever with probability 1.

Consider the case when $0 < c < 1$. Then

$$p_n = p_{n-1}((1-c)p+c) + q_{n-1}((1-c)p) = cp_{n-1} + (1-c)p$$

so that,

$$p_n = Ac^{n-1} p_1 + \frac{(1-c)p}{(1-c)}; \quad p = p_1; \quad A \text{ is a constant}$$

i.e., the probability that the event occurs is the same for all the trials.

$$E(X_n) = p; \quad E(X_{n-1}X_n) = ((1-c)p + c)p$$

$$\Rightarrow \text{Cov}(X_{n-1}, X_n) = cp(1-p); \quad n \geq 1$$

$$\text{Var}(X_n) = p - p^2 = p(1-p)$$

$$\Rightarrow \text{Cor}(X_{n-1}, X_n) = \frac{cp(1-p)}{p(1-p)} = c$$

Now,

$$\begin{aligned} E(X_{n-2}, X_n) &= p_{11}^{(2)} p = (1-c)^2 (1-p)p^2 + ((1-c)p + c)^2 p \\ &= p(1-c)^2 ((1-p)p + p^2) + 2cp^2(1-c) + c^2 p \\ &= (1-c)^2 p^2 + 2cp^2(1-c) + c^2 p \\ &= p^2(1-c^2) + c^2 p \\ &= p^2 + c^2(1-p)p \\ \Rightarrow \text{Cov}(X_{n-2}, X_n) &= c^2(1-p)p \end{aligned}$$

and in general,

$$\text{Cov}(X_{n-k}, X_n) = c^k p(1-p)$$

$$\Rightarrow \text{Cor}(X_{n-k}, X_n) = c^k; \quad k \geq 1$$

Now, consider the accumulated number of successes in n trials in the Markov-Bernoulli

sequence, i.e. $S_n = X_1 + X_2 + \dots + X_n$

$$E(S_n) = np$$

$$\begin{aligned} \text{Var}(S_n) &= \sum_{k=1}^n \text{Var}(X_k) + 2 \sum_{\substack{j < k \\ k=1}}^n \text{Cov}(X_j, X_k) \\ &= np(1-p) + 2p(1-p) \frac{c}{1-c} \left((n-1) - c \frac{1-c^{n-1}}{1-c} \right) \end{aligned}$$

If for $n \rightarrow \infty$, and $p \rightarrow 0, np \rightarrow \lambda$, then

$$E(S_n) \rightarrow \lambda$$

$$\text{Var}(S_n) \rightarrow \lambda + \frac{2\lambda c}{1-c}$$

If $c = 0$, then $\text{Var}(S_n) \rightarrow \lambda$, i.e. for a sequence of independent Bernoulli trials, the limiting distribution is Poisson.

(iv) Correlated random walk:

Consider a sequence of random variables $X_n, n = 0, 1, 2 \dots$ such that each of X_n assumes only two values -1 and 1 with t.p.m.

$$P = \begin{matrix} & X_{n-1} & -1 & 1 \\ & X_n & & \\ -1 & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} & & \end{matrix}; \quad 0 < a, b < 1$$

with the initial distribution $P(X_0 = 1) = p_1 = 1 - P(X_0 = -1) = 1 - q_1$

Let X_n denote the direction of the movement (to the right (left) corresponding to the value $1(-1)$ at the n^{th} step. Let

$$p_n = P(X_n = 1)$$

$$q_n = P(X_n = -1) = 1 - p_n$$

$$\Rightarrow p_n = (1-a-b)p_{n-1} + a, \quad n \geq 1$$

$$= \frac{a}{a+b} + \left(p_1 - \frac{a}{a+b} \right) (1-a-b)^n$$

Now,

$$E(X_n) = p_n - (1 - p_n) = 2p_n - 1$$

$$\text{Var}(X_n) = p_n + (1 - p_n) - (2p_n - 1)^2 = 1 - (2p_n - 1)^2$$

$$\begin{aligned} E(X_{n-1}X_n) &= 1((1-b)p_{n-1} + (1-a)q_{n-1}) - 1(bp_{n-1} + aq_{n-1}) \\ &= (1-2a) + 2(a-b)p_{n-1} \end{aligned}$$

$$\Rightarrow \text{Cov}(X_{n-1}X_n) = (1-2a) + 2(a-b)p_{n-1} - (2p_n - 1)(2p_{n-1} - 1)$$

Put $1-2a = c$, $a = b$

$$\Rightarrow p_n = \frac{1}{2} + \left(p_1 - \frac{1}{2}\right)c^n = \frac{1}{2}(1 + (2p_1 - 1)c^n)$$

$$E(X_n) = (2p_1 - 1)c^n$$

$$\text{Var}(X_n) = 1 - (2p_1 - 1)^2 c^{2n}$$

$$E(X_{n-1}X_n) = c$$

$$\text{Cov}(X_{n-1}X_n) = \begin{cases} c - ((2p_1 - 1)c^n)((2p_1 - 1)c^{n-1}), & \text{if } p_1 \neq \frac{1}{2} \\ c, & \text{if } p_1 = \frac{1}{2} \end{cases}$$

$$\text{Cor}(X_{n-1}X_n) = \begin{cases} c - c^{2n-1}(2p_1 - 1)^2, & \text{if } p_1 \neq \frac{1}{2} \\ c, & \text{if } p_1 = \frac{1}{2} \end{cases}$$

Problems

1. In the gambler's ruin problem, show that

$P(\text{Gambler A wins the next game} | \text{he has amount } i, \text{ he is the final winner})$

$$= \begin{cases} \frac{p \left(1 - \frac{q}{p}\right)^{i+1}}{1 - \left(\frac{q}{p}\right)^i}, & \text{if } p_1 \neq \frac{1}{2} \\ \frac{i+1}{2i}, & \text{if } p_1 = \frac{1}{2} \end{cases}$$

2. For a Markov Chain, show that

$$p_{ij}^{(n)} \geq \sum_{k=0}^n f_{ij}^{(k)} p_{jj}^{(n-k)}$$

3. If $f_{ii} < 1$ and $f_{jj} < 1$, then show that

$$f_{ij} = \frac{\sum_{n=1}^{\infty} p_{ij}^{(n)}}{1 + \sum_{n=1}^{\infty} p_{jj}^{(n)}}$$

4. A coin is tossed, p being the probability of a head in a toss. Let $\{X_n, n \geq 1\}$ have two states 0 and 1 according as the accumulated number of heads and tails in n tosses are equal or unequal. Show that the states are transient when $p \neq \frac{1}{2}$ and persistent null when $p = \frac{1}{2}$.