Chapter 2

Markov Chains - I

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Key words: Markov chains, conditional probability, transition probabilities, transition probability matrix, initial distribution, order of a chain, state space of a chain, random walks.

Suggested readings:

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2.1 Introduction

Consider a simple coin-tossing experiment repeated for a number of times. The possible outcomes at each trial are two: a head with probability, say, p and a tail with probability q (= 1-p). Define a random variable associated with the n^{th} trial as

$$X_n = \begin{cases} 1, & \text{if a head appears at the } n^{th} \text{ toss} \\ 0, & \text{if a tail appears at the } n^{th} \text{ toss} \end{cases}$$

$$\Rightarrow \text{ for } n = 1, 2, 3, \ldots$$

$$P(X_n = 1) = p = 1 - P(X_n = 0) = 1 - q$$

then, we have a sequence of random variables $X_1, X_2, \ldots$ which are independent, i.e., outcome of the n^{th} trial does not depend in any manner on the previous trials numbered 1,2,…,n-1.

Now, consider the random variable given by the partial sum

$$S_n = X_1 + X_2 + \ldots + X_n$$

i.e., accumulated number of heads in first n trials. The possible values which S_n can take are 0,1,2,…,n and $S_{n+1} = S_n + X_{n+1}$. Given $S_n = j$, S_{n+1} can take only two values

$$S_{n+1} = \begin{cases} j & \text{with probability } q \\ j+1 & \text{with probability } p \end{cases} \qquad j = 0, 1, 2, \ldots, n$$

and for given S_n, these values are not at all affected by the values of the variables $S_1, S_2, \ldots, S_{n-1}$, i.e.,

$$P(S_{n+1} = j + 1 \mid S_n = j) = p$$

and

$$P(S_{n+1} = j \mid S_n = j) = q$$

The sequence $\{S_1, S_2, ..., S_{n-1}\}$ of random variables is a Markov chain.

Now, the idea of Markov chains is fairly clear. Those stochastic processes for which the outcome of immediate future is a function of present outcome only and not a function of previous outcomes is a Markov chain. We define a Markov chain as follows:

Definition: A stochastic process $\{X_n, n = 0,1,2\ldots\}$ is called a Markov chain, if, for $j, k, j_1, ..., j_{n-1} \in \mathrm{N}$, N is the set of Natural numbers (or any subset of the set of integers I)

$$P(X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, ..., X_0 = j_{n-1}) = P(X_n = k \mid X_{n-1} = j) = p_{jk} \text{ (say)}$$

The outcomes are called the states of the Markov chain. p_{jk} is the conditional probability of moving from state j to state k and is known as the transition probability of transition from state j to state k. Being a function of both states j and k, p_{jk} is not fixed for any state k.

Examples:

(1) **Polya's urn model:**

Recall the Polya's urn model considered in chapter 1 where we had defined some stochastic processes with the help of these models. However, these processes were not Markov processes.

Now, consider the random variable Y_n, which gives the number of black balls at the time of the n^{th} drawing. Then Y_1 takes only one value b, Y_2 takes values b and $b + c$, Y_3 takes values b, $b + c$, $b + 2c$, and, in general, Y_n assumes n values b, $b + c$, $b + 2c$,… $b + (n\text{-}1)c$.

Given Y_{n-1} assumes one of the possible values $i = b + kc$, $k = 0, 1, 2, ..., n-2$, then Y_n assumes values $j = i$ and $j = i + c$ with the probabilities

$$P(Y_n = i \mid Y_{n-1} = i) = a_1 = P(X_{n-1} = 0) = \frac{r}{b + r}$$

and $$P(Y_n = i + c \mid Y_{n-1} = i) \;=\; b_1 \;=\; P(X_{n-1} = 1) \;=\; \frac{b}{b+r}$$

Then, $\{Y_n , n \geq 1\}$ is a Markov chain.

2.2 Specifications of Markov chains

Homogenous Markov chains: A Markov chain $\{X_n , n \geq 0\}$is a homogenous Markov chain if the transition probability p_{jk} is independent of n, i.e., the chain has stationary probabilities.

If p_{jk} depends on then n, then the chain is a non-homogenous Markov chain. In this case the transition probabilities are denoted by ${}^{(n)}p_{jk}$.

The transition probabilities satisfy the following relationship

$$p_{jk} \;\geq\; 0; \;\; \sum_k p_{jk} \;=\; 1 \;\; \forall j, \;\; j, k \in S$$

S is the state space of the Markov chain

m-step transition probabilities: If the transition from state j to state k involves more than one step, i.e., for a pair of states (j ,k), state j being the state of the system at the n^{th} trial, it takes m (>1) steps to reach the system to state k , then the transition probability is an m-step transition probability and is denoted by $p_{jk}^{(m)}$. We write

$$p_{jk}^{(m)} \;=\; P(X_{n+m} = k \mid X_n = j)$$

$p_{jk}^{(1)} \;=\; p_{jk}$ is one-step or unit step transition probability.

Transition matrix: The matrix having the transition probabilities of a Markov chain as its elements is a transition matrix or a transition probability matrix (t.p.m.), i.e., if

$$P \;=\; \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ p_{31} & p_{32} & \cdots \end{pmatrix}$$

then, P is the transition probability matrix of the Markov chain. The $(i+1)^{st}$ row of P is the probability distribution of the values of X_n under the condition that $X_{n-1} = i$. In general, P is a square matrix. Since $p_{jk} \geq 0$; and $\sum_k p_{jk} = 1 \ \forall j$, i.e., the unit row sums, so P is a stochastic matrix. If columns also add up to unity, then P is a doubly stochastic matrix.

Regular matrix: A stochastic matrix P is a regular matrix if all the members of P or some power P^m ($m > 0$) of it, are positive, i.e., if

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ p_{31} & p_{32} & \cdots \end{pmatrix}$$

Then, either $p_{jk} > 0$; or if some $p_{jk} = 0$; then

$$P^m = \begin{pmatrix} p_{11}^{(m)} & p_{12}^{(m)} & \cdots \\ p_{21}^{(m)} & p_{22}^{(m)} & \cdots \\ p_{31}^{(m)} & p_{32}^{(m)} & \cdots \end{pmatrix}; \quad p_{ij}^{(m)} > 0 \quad \forall i, j \in S$$

2.3 Probability distribution associated with a Markov chain

Theorem 2.1: A Markov chain $\{X_n, n \geq 1\}$ is completely specified by the transition matrix P and the initial distribution of X_0, i.e., the probability distribution of $X_r, X_{r+1}, ..., X_{r+n}$ can be obtained in terms of t.p.m. P and the initial distribution of X_r.

Proof: Without any loss of generality, we assume that $r = 0$. Then

$$\begin{aligned} & P(X_0 = a_0, X_1 = a_1, ..., X_{n-2} = i, X_{n-1} = j, X_n = k) \\ & \quad = P(X_n = k \mid X_{n-1} = j, X_{n-2} = i, ..., X_1 = a_1, X_0 = a_0) P(X_{n-1} = j, X_{n-2} = i, ..., X_1 = a_1, X_0 = a_0) \end{aligned}$$

$$= P(X_n = k \mid X_{n-1} = j)P(X_{n-1} = j \mid X_{n-2} = i)P(X_{n-2} = i, ..., X_1 = a_1, X_0 = a_0)$$

$$= P(X_n = k \mid X_{n-1} = j)P(X_{n-1} = j \mid X_{n-2} = i)...P(X_1 = a_1 \mid X_0 = a_0)P(X_0 = a_0)$$

$$= P(X_0 = a_0) p_{a_0 a_1} ... p_{ij} p_{jk}$$

In general,

$$P(X_r = a_r, X_{r+1} = a_{r+1}, ..., X_{r+n-2} = i, X_{r+n-1} = j, X_{r+n} = k) = P(X_r = a_r) p_{a_r a_{r+1}} ... p_{ij} p_{jk}$$

Thus the transition probability matrix along with the initial distribution completely specifies a Markov chain $\{X_n , n \geq 0\}$.

The following result provides a sufficient condition for the existence of a Markov chain .

General existence theorem of Markov chains:

Given the set N and the sequence of stochastic matrices $({}^{(n)}p_{jk})$ or ${}^{(n)}P$, there exists a Markov chain $\{X_n , n \in N\}$ with state space N and the transition probability matrix ${}^{(n)}P$.

2.4 Order of a Markov chain

A Markov chain $\{X_n , n \in N\}$is said to be of order s (s = 1,2,3,…) if for all n,

$$P(X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, ..., X_{n-s} = j_{s-1}, ...) = P(X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, ..., X_{n-s} = j_{s-1})$$

whenever the left hand side is defined.

A Markov chain of order 1, i.e.,

$$P(X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, ..., X_{n-s} = j_{s-1}, ...) = P(X_n = k \mid X_{n-1} = j) = p_{jk}$$

is simply called a Markov chain.

If $p_{jk} = p_k \quad \forall j$ then X_n and X_{n-1} are independent and the chain is said to be of order 0.

A t.p.m. can be non-square also. Let the state that a day is a rainy day is 1 and that it is not rainy is 0. We define

$p_{ijk} = P$ (Today is in state k | yesterday was in state j and the day before yesterday was in state i), $i, j, k = 0,1$.

Then we have a Markov chain of order 2 with t.p.m.

$$P = \begin{pmatrix} p_{000} & p_{010} & p_{001} & p_{011} \\ p_{100} & p_{110} & p_{101} & p_{111} \end{pmatrix}$$

P is not a stochastic matrix as it is non-square.

2.5 Examples

(2) **Random walks:**

Random walks are an essential tool in study of stochastic processes. A random walk constitutes of a particle positioned at a particular position at the beginning of the experiment and is allowed to go to any possible direction with certain non-zero probability. The movement of the particle may be restricted at certain points in its path or the particle may be free to move without any restrictions. Then, depending upon the number of dimensions in which the particle can move or whether the particle is moving along a restricted or unrestricted path, various random walk models may be defined. These random walks have applications in several fields, some of which will be discussed in this course.

2(a) **Random walk with (absorbing) barriers:**

A particle performs a random walk with absorbing barriers, say, at 0 and 4 when walking along a line, i.e., the walk is restricted to one dimension. Whenever it is at a position r $(0 < r < 4)$, it moves to $r+1$ with probability p or to r-1 with probability q, $p + q = 1$. But as soon as it reaches 0 or 4, it remains there itself and gets absorbed. Let X_n be the position of the particle after n steps. The different states of X_n are different positions of the particle. $\{X_n, n \geq 1\}$ is a Markov chain with unit step transition probability

$$\left.\begin{array}{l} P(X_n = r+1 \mid X_{n-1} = r) = p \\ P(X_n = r-1 \mid X_{n-1} = r) = q \end{array}\right\} \quad 0 < r < 4$$

$$P(X_n = 0 \mid X_{n-1} = 0) = 1 = P(X_n = 4 \mid X_{n-1} = 4)$$

The transition probability matrix is given by

$$\begin{array}{ccc} & & \text{states of } X_n \\ \text{states of } X_{n-1} & & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ P \quad = & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

2(b) The random walk in the above example is a simple one since the particle can move to the adjacent positions to its current position and once it reaches some particular states (0,4 in this case) it is not able to move any further. A general situation can be that after reaching states 0 and 4, the particles stays there with a certain non-zero probability and moves to the adjacent position with the complementary probability. This is the general model of random walk with barriers. If a is the probability with which the particle stays at 0 and moves to 1 with probability 1-a and similarly b and 1-b are the probabilities with which the particle stays at 4 or moves to 3 respectively, then the t.p.m. of the Markov chain $\{X_n\}$ with state space $S = \{0,1,2,\ldots,k\ (\geq 1)\}$ is given by

$$\begin{array}{ccc} & & \text{states of } X_n \\ \text{states of } X_{n-1} & & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & k-1 & k \end{array} \\ P_1 \quad = & \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ k-1 \\ k \end{array} & \begin{pmatrix} a & 1-a & 0 & \cdots & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & q & \cdots & 0 & p \\ 0 & 0 & 0 & \cdots & 1-b & b \end{pmatrix} \end{array}$$

If a (b) = 1, then 0 (4) is an absorbing barrier. For a (b) = 0, 0 (4,4) is a reflecting barrier and for 0 < a (b) < 1, 0 (4) is an elastic barrier.

An alternative representation of the above model is the gambler's ruin model. Suppose that a gambler A is playing a game with initial capital $X_0 = x_0$ against an opponent B with initial capital $Y_0 = a\text{-}x_0$, the stake being unity each time .Let X_n is the fortune of gambler A after n games. He wins a game with probability p thus increasing his fortune by a unit amount or looses a game with probability q thus decreasing his fortune by a unit amount. If his fortune reduces to 0, he will be ruined and if his fortune reaches a, his opponent B will be ruined. Then, P is the t.p.m. of the gambler's ruin .

Now, suppose that once the gambler reaches state 0, his adversary is kind enough to lend him a unit amount so that the game doesn't stop and r is the probability that a game results in a draw then the game will never stop and the t.p.m. is now given by

$$P_2 = \begin{pmatrix} a & 1-a & 0 & \cdots & 0 & 0 \\ q & r & p & \cdots & 0 & 0 \\ 0 & q & r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & q & \cdots & r & p \\ 0 & 0 & 0 & \cdots & 1-b & b \end{pmatrix}$$

i.e., if 0 and a are both reflecting barriers, then the game would never stop.

2(c) **Cyclical random walks:** In this case, the possible states of the system are arranged in a cyclic manner so that no state lies at the extreme and every state has adjacent states both to the right and to the left. The transition is possible only to the adjacent states. Then, the t.p.m. is given by

$$P_3 = \begin{pmatrix} 0 & p & 0 & \cdots & 0 & q \\ q & 0 & p & \cdots & 0 & 0 \\ 0 & q & r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & q & \cdots & 0 & p \\ p & 0 & 0 & \cdots & q & 0 \end{pmatrix}$$

In general, if it is possible to move from any state to any state arranged cyclically, then

$$P_4 = \begin{pmatrix} p_0 & p_1 & p_2 \cdots & p_{k-2} & p_{k-1} \\ p_{k-1} & p_0 & p_1 \cdots & p_{k-3} & p_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_2 & p_3 & p_4 \cdots & p_0 & p_1 \\ p_1 & p_2 & p_3 \cdots & p_{k-1} & p_0 \end{pmatrix}$$

(3) **The Ehrenfest model of diffusion:** Consider two urns *A* and *B* having *k* balls in all. A ball is selected at random from either of the urns and is transferred to the other urn. The system is in state *r* at a certain time point if the urn *A* has exactly *r* balls in it. The system passes to states *r*-1 or *r*+1 according as the next ball is drawn from urn *A* or from the urn *B* respectively. If X_n is the random variable denoting the state of the system at the n^{th} trial, then

$$p_{ij} = P(X_n = j \mid X_{n-1} = i) = \begin{cases} 1-\dfrac{i}{k+1}, & j = i+1 \\ \dfrac{i}{k+1}, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

Putting $\dfrac{1}{k+1} = \rho$, the t.p.m. is given by

$$P = \begin{array}{c} \\ \text{states of } X_{n-1} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k-1 \\ k \end{array} \begin{array}{c} \text{states of } X_n \\ \begin{array}{cccccc} 0 & 1 & 2 & 3\cdots & k-1 & k \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0\cdots & 0 & 0 \\ \rho & 0 & 1-\rho & 0\cdots & 0 & 0 \\ 0 & 2\rho & 0 & 1-2\rho\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & 0 & 1-(k-1)\rho \\ 0 & 0 & 0 & 0\cdots & 1 & 0 \end{pmatrix} \end{array}$$

In terms of the random walk, the situation may be described as follows: From position *r* initially, the particle moves to left or right with probabilities $r\rho$ or 1- $r\rho$ respectively. The particle is more likely to move to the left or to the right according as $n < \dfrac{k}{2}$ or $n > \dfrac{k}{2}$.

(4) **Bernoulli-Laplace diffusion model:** Two urns *A* and *B* have 2*k* balls, out of which *k* are black balls and remaining *k* are red balls. The system is in state *r* if urn *A* contains *r* black balls. A ball is selected at random from either of the urns (say, *A*) and is passed to the other urn. The matter of the two urns is incompressible so a ball from urn *B* is also passed to the urn *A*. The total number of balls in each urn at any trial is the same, i.e., *k*. Simply speaking, at each trial one ball is picked from each of the urns and transferred to the other. Then, denoting the state of the system at the n^{th} trial by X_n, we have

$$p_{ij} = P(X_n = j \mid X_{n-1} = i) = \begin{cases} \left(\dfrac{i}{k}\right)^2, & j = i-1 \\ \left(\dfrac{k-i}{k}\right)^2, & j = i+1 \\ \dfrac{2i(k-i)}{k^2}, & j = i \\ 0, & \text{otherwise} \end{cases}$$

Putting $\frac{i}{k} = i\rho$, the t.p.m. is given by

$$P = \begin{array}{c|c} & \text{states of } X_n \\ \text{states of } X_{n-1} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & k-1 & k \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ k \end{array} & \begin{pmatrix} 0 & (k-1)^2\rho^2 & 0 & \cdots & 0 & 0 \\ \rho^2 & 2(k-1)\rho & (k-1)^2\rho^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)^2\rho^2 & 0 \end{pmatrix} \end{array}$$

(5) **Success runs of length *k*:** A run in an experiment is an uninterrupted sequence of symbols (or outcomes) of the same type. The concept of runs has been used to define **Distributions of order *k*** and has found applications in fields such as Statistical Quality Control, Statistical Process Control, Pattern Recognition etc. We describe the concept of a run as follows:

Let a coin is tossed indefinitely with *p* being the probability of a head. Define a random variable X_n as

$$X_n = k, \text{ if } n^{th} \text{ trial results in } k^{th} \text{ consecutive success}, \; k = 0, 1, 2, ..., n$$

Then $\{X_n\}$ is a Markov chain with unit-step transition probabilities

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} p, & k = j+1 \\ q, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

and the t.p.m. P is given by

$$P = \begin{array}{c} \\ \text{states of } X_{n-1} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \end{array} \begin{array}{c} \text{states of } X_n \\ \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & k & k+1 & \cdots \end{array} \\ \begin{pmatrix} q & p & 0 & \cdots & 0 & 0 & \cdots \\ q & 0 & p & \cdots & 0 & 0 & \cdots \\ q & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ q & 0 & 0 & \cdots & 0 & p & \cdots \\ q & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \end{pmatrix} \end{array}$$

In terms of Polya's urn models, suppose that drawing, with replacement and without addition of any more balls, of a black ball is a success. Then, P is t.p.m. of the event of k consecutive black balls, the probability of a single black drawing being $p = \dfrac{b}{b+r}$.

(6) **Partial sums of independent random variables:** We recall the random variable Y_n defined in Polya's urn model, which was the number of black balls at the time of the n^{th} draw. Clearly, $Y_n = X_1 + X_2 + \ldots + X_n$, the possible values of Y_n being k, $k = 0,1,2,\ldots n$. Then Y_n, the accumulated number of black balls in n draws is the partial sum of independent random variables X_n, $n = 0,1,2,\ldots$ and $\{Y_n, n \geq 0\}$is a Markov chain with t.p.m.

$$\begin{array}{c} \text{states of } Y_n \\ \text{states of } Y_{n-1} \quad 0 \quad 1 \quad 2 \quad \cdots \quad k \quad k+1 \quad \cdots \end{array}$$

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \end{array} \begin{pmatrix} q & p & 0 \cdots 0 & 0 \cdots \\ 0 & q & p \cdots 0 & 0 \cdots \\ 0 & 0 & q \cdots 0 & 0 \cdots \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0 \cdots q & p \cdots \\ 0 & 0 & 0 \cdots 0 & q \cdots \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \end{pmatrix}$$

This process is an **additive process**.

In terms of the random walk, the walk is restricted, 0 being the absorbing barrier. If, however, we modify the variable X_n as

$$X_n = \begin{cases} 1, & \text{if a black ball appears at the } n^{th} \text{ trial} \\ -1, & \text{if a red ball appears at the } n^{th} \text{ trial} \end{cases}$$

then, the t.p.m. is given by

$$\begin{array}{c} \text{states of } Y_n \\ \text{states of } Y_{n-1} \cdots \; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \; \cdots \end{array}$$

$$P = \begin{array}{c} \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ q & p & 0 \cdots 0 & 0 \cdots \\ 0 & q & p \cdots 0 & 0 \cdots \\ 0 & 0 & q \cdots 0 & 0 \cdots \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0 \cdots q & p \cdots \\ 0 & 0 & 0 \cdots 0 & q \cdots \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \end{pmatrix}$$

(7) **A queuing model -(M/M/1) queue:** Consider a counter to which customers arrive for service. There is a server who serves one customer (if any present) only at time points 0,1,2… Assume that in the time interval $(n, n+1)$, a number Y_n of customers arrives, where Y_n are i.i.d. random variables with mass function

$$P(Y_n = k) = p_k, \; k = 0,1,2,...; \; n = 0,1,2,...$$

The service station has a capacity of at most c customers including the one being served and further arrivals are not entertained by the service station (lost customers). Then $\{X_n,\ n \geq 1\}$, the number of customers at time point n is a Markov chain with state space $S = \{0,1,2,\ldots,c\}$. We have

$$X_{n+1} = \begin{cases} Y_n, & if\ X_n = 0 \text{ and } 0 \leq Y_n \leq c-1 \\ X_n + Y_n, & if\ 1 \leq X_n \leq c \text{ and } 0 \leq Y_n \leq c+1-X_n \\ c, & \text{otherwise} \end{cases}$$

Then, the t.p.m. is given by

$$P = \begin{array}{c|c} & \text{states of } X_n \\ \text{states of } X_{n-1} & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & c-2 & c-1 & c \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \\ c-1 \\ c \end{array} & \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{c-2} & p_{c-1} & \sum_{i\geq 0} p_{c+i} \\ p_0 & p_1 & p_2 & \cdots & p_{c-2} & p_{c-1} & \sum_{i\geq 0} p_{c+i} \\ 0 & p_0 & p_1 & \cdots & p_{c-3} & p_{c-2} & \sum_{i\geq 0} p_{c+i-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_0 & p_1 & \sum_{i\geq 0} p_{2+i} \\ 0 & 0 & 0 & \cdots & 0 & p_0 & \sum_{i\geq 0} p_{1+i} \end{pmatrix} \end{array}$$

If the service station can accommodate an infinite number of customers, then the t.p.m. will be given by

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{c-2} & p_{c-1} & \cdots \\ p_0 & p_1 & p_2 & \cdots & p_{c-2} & p_{c-1} & \cdots \\ 0 & p_0 & p_1 & \cdots & p_{c-3} & p_{c-2} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & p_0 & \cdots \end{pmatrix}$$

If arrival of customers is a Poisson variable and the service time distribution is exponential, then the above model is an $(M/M/1)$ queue.

(8) Consider a Markov chain $\{X_n, n = 1,2,3\ldots\}$ with $S = \{1,2,3\}$ and having the t.p.m.

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \end{array}$$

and the initial distribution (0.7,0.2,0.1). Then,

$$\begin{aligned} & P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) \\ &= P(X_3 = 2 \mid X_2 = 3)P(X_2 = 3 \mid X_1 = 3)P(X_1 = 3 \mid X_0 = 2)P(X_0 = 2) \\ &= p_{32}p_{33}p_{23}P(X_0 = 2) \\ &= (0.2)(0.3)(0.4)(0.2) \\ &= .0048 \end{aligned}$$

(9) **Cell genetics:** It is a well-known fact that the cells of an organism divide to form new cells. The new cells, known as the daughter cells, are either identical to the parent cell or are differentiated cells. Consider the cells of a primitive organism, which have *N* molecules of two types only, say, *A* and *B*. The cell is in state *r* if it has *r* molecules of type *A* and *N* - *r* molecules of type *B*. Prior to cell division, molecules of each type get doubled. The daughter cell inherits *N* molecules chosen at random from 2*r* molecules of type *A* and 2*N* -2*r* molecules of type *B*. Then the probability that the daughter cell is in state *k* is given by

$$p_{rk} = \frac{\binom{2r}{k}\binom{2N-2r}{N-k}}{\binom{2N}{N}}; \quad 0 \le r, k \le N$$

In this case, 0 and 2*N* are the absorbing states.

(10) **Cell genetics** (contd.)**:** Now, consider a simple haploid model in which there is a fixed population of genes, which is of size 2*N*. The population is composed of two genes, type- *a* and type- *A*. The

population enters in the next generation by random reproduction, where all the $2N$ genes are selected randomly in independent Binomial trials with the following probability schedule:

$$p_j = \frac{j}{2N}; \; q_j = 1 - \frac{j}{2N}$$

where j is the number of type-a genes in the population of $2N$ genes, i.e., a newly selected gene will be a type-a gene with probability p_j, and a type-A gene with probability q_j. The repeated selections are done with replacement.

Define a random variable X_n as the number of type-a genes in the nth generation. Then $\{X_n, n \geq 1\}$ is a Markov chain with the state space S = {0,1,2…$2N$} and the transition probabilities

$$p_{ik}^{(n)} = P(X_{n+1} = k \mid X_n = i) = \binom{2N}{k} p_j^k q_j^{2N-k}$$

0 and 2N are absorbing states.

Now, suppose that the genes are subject to chance mutation prior to division. Let, at each stage, the probability of $a \to A$ mutation is α, whereas the probability of $A \to a$ is β. Then

$$p'_j = \frac{j}{2N}(1-\alpha) + \left(1 - \frac{j}{2N}\right)\beta;$$
$$q'_j = \frac{j}{2N}\alpha + \left(1 - \frac{j}{2N}\right)(1-\beta)$$

and the transition probability is

$$p_{ik}^{(n)} = P(X_{n+1} = k \mid X_n = i) = \binom{2N}{k} p'^k_j \, q'^{2N-k}_j .$$

2.6 Finite Markov chains

A Markov Chain $\{X_n, n \geq 1\}$ with state space S is said to be finite if $S = \{s_1, s_2, ..., s_k, \; k < \infty\}$., i.e., the state space of the chain is finite. In this case, the transition probability matrix of the chain is a square matrix of order k.

If S takes values on the set if integers, i.e., S = {… -2, -1, 0, 1, 2, 3…}, the chain is denumerably infinite or denumerable Markov chain.

Higher transition probabilities: Till now, we have come across unit step transitions and all the transitions involving higher transitions are to be expressed in terms of unit step transitions. Now, we find an expression for higher transitions.

For a homogenous Markov chain, $p_{jk}^{(n)} = P(X_{n+m} = k \mid X_n = j)$ is the probability that the system, which is in state j at the n^{th} trial, will reach state k after m steps in $(n + m)^{\text{th}}$ trial. We wish to obtain an expression for $p_{jk}^{(n)}$.

Consider $m = 2$.

$$p_{jk}^{(2)} = P(X_{n+2} = k \mid X_n = j)$$

$$= P(X_{n+2} = k \mid X_{n+1} = r, X_n = j) P(X_{n+1} = r \mid X_n = j)$$

for some intermediate fixed state r

$$= P(X_{n+2} = k \mid X_{n+1} = r) P(X_{n+1} = r \mid X_n = j)$$

$$= p_{rk} p_{jr} \qquad r = 1, 2, 3...$$

Since different values of r are mutually exclusive so

$$p_{jk}^{(2)} = P(X_{n+2} = k \mid X_n = j)$$

$$= \sum_r P(X_{n+2} = k, X_{n+1} = r \mid X_n = r)$$

$$= \sum_r p_{rk} p_{jr}$$

Similarly,

$$p_{jk}^{(3)} = P(X_{n+3} = k \mid X_n = j)$$

$$= \sum_r P(X_{n+3} = k, X_{n+2} = r \mid X_n = j) = \sum_r p_{rk}^{(2)} p_{jr}$$

Again

$$p_{jk}^{(3)} = P(X_{n+3} = k \mid X_n = j)$$

$$= \sum_r P(X_{n+3} = k, X_{n+1} = r \mid X_n = j) = \sum_r p_{rk}\, p_{jr}^{(2)}$$

$$\Rightarrow \quad p_{jk}^{(3)} = \sum_r p_{rk}\, p_{jr}^{(2)} = \sum_r p_{rk}^{(2)}\, p_{jr}$$

Continuing the process, by induction, we have

$$p_{jk}^{(m+n)} = \sum_r p_{jr}^{(m)} p_{rk}^{(n)} = \sum_r p_{jr}^{(n)} p_{rk}^{(m)}; \quad m, n, r > 0 \tag{2.1}$$

It can be seen that

$$p_{jk}^{(m+n)} \geq p_{jr}^{(n)} p_{rk}^{(m)}; \quad m, n, r > 0$$

Equation (2.1) is a particular case of Chapman - Kolmogorov equation and is always satisfied by the transition probability matrix of a Markov chain.

The results can be neatly presented in matrix form. If $P = (p_{ij})$ is the t.p.m. of the Markov chain, and $P^{(m)} = \left(p_{ij}^{(m)}\right)$ the m-step t.p.m., then for $m = 2$, we have

$$P^{(2)} = \left(\sum_r p_{rk}\, p_{jr}\right) = \left(p_{rk}\right)\left(p_{jr}\right) = P.P = P^2$$

Similarly, $$P^{(m+1)} = P^{(m)}.P = P^m.P = P^{m+1}$$

and $$P^{(m+n)} = P^{(m)}.P^{(n)} = P^m.P^n = P^{m+n}$$

2.7 Examples

(11) In the above example with t.p.m.

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \end{array}$$

and the initial distribution (0.7,0.2,0.1) , suppose we want to find out P(X_2 = 3).

Then

$$P(X_2 = 3) = \sum_i P(X_2 = 3 \mid X_0 = i) P(X_0 = i)$$

$$P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

$$\Rightarrow P(X_2 = 3) = (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$$

$$= 0.279$$

(12) A magician has arranged m people in a circle who have been given a ring, which can be passed, to either of the persons sitting on the left or to the right of the person who is having it presently, with equal probability. All the m individuals have been numbered from 1 to m. The magician is blindfolded. The claim of the magician is that he can tell with certain non-zero probability about the location of the ring if he is told about the location when the game began. As a statistician you know that the magician is relying upon the concept of Markov chains. Help him to calculate the t.p.m. If m = 2, find the probability that after two passes the ring will be back to the person who started the game.

Sol: Let X_0 be the individual who had the ring when the game began and X_n is the individual who had the ring after n passes. Then the states of X_n are 1,2,3…m and therefore,

$$P(X_0 = i) = \frac{1}{m}$$

$$P(X_n = i \mid X_{n-1} = i) = 0$$

$$P(X_n = j \mid X_{n-1} = i) = \frac{1}{m-1}, \quad i \neq j$$

The t.p.m. of the chain is

$$
\begin{array}{c} \\ \\ P_1 \;=\; \end{array}
\begin{array}{c|c}
 & \text{states of } X_n \\
\text{states of } X_{n-1} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & m-1 & m \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ m-1 \\ m \end{array} &
\begin{pmatrix}
0 & \frac{1}{m-1} & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\
\frac{1}{m-1} & 0 & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\
\frac{1}{m-1} & \frac{1}{m-1} & 0 & \cdots & \frac{1}{m-1} \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{1}{m-1} & \frac{1}{m-1} & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\
\frac{1}{m-1} & \frac{1}{m-1} & \frac{1}{m-1} & \cdots & 0
\end{pmatrix}
\end{array}
$$

For m = 3, P reduces to

$$
P \;=\; \begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}
\end{array}
$$

and

$$
P^2 \;=\; \begin{array}{c c}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}
\end{array}
$$

Hence $P(X_2 = i \mid X_0 = i) = 0.5; \;\; i = 1, 2, 3$

(13) In Polya's urn models, consider a sequence $\{X_n,\ n \geq 2\}$ of independent trials with probability p of a success (a black ball, out of b black and r red balls). Denote the states of X_n by states 1,2,3,4 according as trials numbers n-1 and n result in SS, SF, SF and FF respectively. Show that $\{X_n,\ n \geq 2\}$ is a Markov chain. Find P and P^m.

Sol:

$$
P \;=\; \begin{array}{c c}
 & \text{states of } X_n \\
\text{states of } X_{n-1} & \begin{array}{cccc} SS & SF & FS & FF \end{array} \\
\begin{array}{c} SS \\ SF \\ FS \\ FF \end{array} &
\begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ p & q & 0 & 0 \\ 0 & 0 & p & q \end{pmatrix}
\end{array}
$$

As the state space and the transition probability matrix of $\{X_n, n \geq 2\}$ is defined so $\{X_n, n \geq 2\}$ is a Markov chain. Now,

$$P^2 = \begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ p & q & 0 & 0 \\ 0 & 0 & p & q \end{pmatrix}\begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ p & q & 0 & 0 \\ 0 & 0 & p & q \end{pmatrix} = \begin{pmatrix} p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \end{pmatrix}\begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ p & q & 0 & 0 \\ 0 & 0 & p & q \end{pmatrix} = \begin{pmatrix} p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \end{pmatrix}$$

and , in general, if

$$P^{\mathrm{m}} = \begin{pmatrix} p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \end{pmatrix}$$

then,

$$P^{\mathrm{m+1}} = P^{\mathrm{m}} = \begin{pmatrix} p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \\ p^2 & pq & pq & q^2 \end{pmatrix}$$

(14) Consider a binary communication system, which transmits signals through the presence or absence of an electrical pulse. The system, when working on the signal, may transmit it as it is with probability q or may convert it to the other form with the probability p. Let the presence of the pulse be denoted by state 1 and its absence by state 0. If X_n , $n \geq 1$be the pulse leaving the n^{th} stage and X_0 be the pulse entering the first stage, then $\{X_n, n \geq 1\}$ is a Markov chain. Find $\lim_{n\to\infty} P^{(n)}$.

Sol: The t.p.m. of the chain is given by

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

We want to study the long-term behaviour of this chain. Now,

$$P^2 = \begin{pmatrix} q & p \\ p & q \end{pmatrix}\begin{pmatrix} q & p \\ p & q \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(q-p)^2 & \frac{1}{2}-\frac{1}{2}(q-p)^2 \\ \frac{1}{2}-\frac{1}{2}(q-p)^2 & \frac{1}{2}+\frac{1}{2}(q-p)^2 \end{pmatrix}$$

And, by induction,

$$P^m = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(q-p)^m & \frac{1}{2}-\frac{1}{2}(q-p)^m \\ \frac{1}{2}-\frac{1}{2}(q-p)^m & \frac{1}{2}+\frac{1}{2}(q-p)^m \end{pmatrix}$$

Since $0 < q, p < 1$, so as $m \to \infty$,

$$\lim_{m\to\infty} p_{00}^{(m)} = \lim_{m\to\infty} p_{01}^{(m)} = \lim_{m\to\infty} p_{10}^{(m)} = \lim_{m\to\infty} p_{11}^{(m)} = \frac{1}{2}$$

i.e.,

$$\lim_{m\to\infty} P^m = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(15) Suppose that a fair die is tossed. Let the states of X_n be k (k = 1, 2, 3, 4, 5, 6) where k is the maximum number shown in the first n tosses. Find P, P^2 and P^n. Calculate P(X_2 = 6).

Sol:

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} \frac{1}{6}, & \text{if } k > j \\ \frac{k}{6}, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$$

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{6} \end{pmatrix}$$

$$P^2 = \left(\frac{1}{6}\right)^2 \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

$$\Rightarrow \; P(X_2 = 6) = \sum_i P(X_2 = 6 \mid X_0 = i)P(X_0 = i) = \frac{1}{6}\left(5\left(\frac{11}{36}\right) + \frac{36}{36}\right) = \frac{91}{216}$$

$$P^3 = P^2\,\mathrm{P} = \left(\frac{1}{6}\right)^3 \begin{pmatrix} 1 & 7 & 19 & 37 & 61 & 91 \\ 0 & 8 & 19 & 37 & 61 & 91 \\ 0 & 0 & 27 & 37 & 61 & 91 \\ 0 & 0 & 0 & 64 & 61 & 91 \\ 0 & 0 & 0 & 0 & 125 & 91 \\ 0 & 0 & 0 & 0 & 0 & 216 \end{pmatrix}$$

And, in general

$$P^n = \left(\frac{1}{6}\right)^n \begin{pmatrix} 1^n & 2^n-1^n & 3^n-2^n & 4^n-3^n & 5^n-4^n & 6^n-5^n \\ 0 & 2^n & 3^n-2^n & 4^n-3^n & 5^n-4^n & 6^n-5^n \\ 0 & 0 & 3^n & 4^n-3^n & 5^n-4^n & 6^n-5^n \\ 0 & 0 & 0 & 4^n & 5^n-4^n & 6^n-5^n \\ 0 & 0 & 0 & 0 & 5^n & 6^n-5^n \\ 0 & 0 & 0 & 0 & 0 & 6^n \end{pmatrix}$$

(16) Suppose that a ball is thrown at random to one of the three cells. Let X_n, $(n \geq 1\}$be a random variable having value k (= 1, 2, 3) if after n throws, k cells are occupied. Find P and P^n.

Sol: Let p be the probability of occupying a blank cell.

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} p, & \text{if } k = j+1 \\ 1-p, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$$

In this case, $p = \dfrac{1}{3}$. Hence

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

and

$$P^2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & \frac{6}{9} & \frac{2}{9} \\ 0 & \frac{4}{9} & \frac{5}{9} \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^3 = \frac{1}{27}\begin{pmatrix} 1 & 6 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 1 & 14 & 12 \\ 0 & 8 & 19 \\ 0 & 0 & 27 \end{pmatrix}$$

And, in general, if

$$P^n = \frac{1}{3^n}\begin{pmatrix} 1^n & 2\left(\sum_{i=0}^{n} 2^i - 1\right) & \left(\sum_{i=0}^{n-1} 2^{i+1} + 2\sum_{i=0}^{n-2} i3^i\right) \\ 0 & 2^n & \sum_{i=0}^{n-1} 2^i 3^{n-i+1} \\ 0 & 0 & 3^n \end{pmatrix}$$

Then,

$$P^{n+1} = \frac{1}{3^{n+1}}\begin{pmatrix} 1^{n+1} & 2\left(\sum_{i=0}^{n+1} 2^i - 1\right) & \left(\sum_{i=0}^{n} 2^{i+1} + 2\sum_{i=0}^{n-1} i3^i\right) \\ 0 & 2^{n+1} & \sum_{i=0}^{n} 2^i 3^{n-i+1} \\ 0 & 0 & 3^{n+1} \end{pmatrix}$$

(17) In the problem above, if a ball is thrown at random to one of the r cells, find P and P^2.

Sol:

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} \dfrac{r-j}{r}, & \text{if } k = j+1 \\ \dfrac{j}{r}, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$P = \begin{pmatrix} \frac{1}{r} & \frac{r-1}{r} & 0\cdots 0 \\ 0 & \frac{2}{r} & \frac{r-2}{r}\cdots 0 \\ \vdots & \vdots & \vdots \cdots \vdots \\ 0 & 0 & 0 \cdots 1 \end{pmatrix}$$

And,

$$P^2 = \frac{1}{r^2}\begin{pmatrix} 1 & r-1 & 0\cdots 0 \\ 0 & 2 & r-2\cdots 0 \\ \vdots & \vdots & \vdots \cdots \vdots \\ 0 & 0 & 0\cdots r \end{pmatrix}\begin{pmatrix} 1 & r-1 & 0\cdots 0 \\ 0 & 2 & r-2\cdots 0 \\ \vdots & \vdots & \vdots \cdots \vdots \\ 0 & 0 & 0\cdots r \end{pmatrix}$$

$$= \frac{1}{r^2}\begin{pmatrix} 1 & 3(r-1) & (r-1)(r-2) & 0 & \cdots 0 \\ 0 & 2^2 & (r-2)^2 & 0 & \cdots 0 \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\ r^2 \end{pmatrix}$$

(18) **Quality control:** As we know, one of the fundamental pillars of any production is the quality of the product and as such quality control has become an essential operation of any production process. This example demonstrates how Markov chains can be used in order to exercise quality control in any production process.

In a production line, every item has probability p of being defective. Every item has its quality of being defective or non-defective independent of the other items. As a measure of quality check, the following inspection plan is adopted:

Initially every item is sampled as it is produced. This process is continued until there is a run of size i of non-defective units. After this run, only one unit out of every r items is sampled randomly until a defective unit is found. When a defective unit is found, after this again 100% inspection is followed till a non-defective run of length i is obtained and the process continues in this manner.

Let X_k (k= 0,1,2…I) be the state of the system where X_k is the random variable denoting the size of the run of non-defective units in the 100% sampling portion of the plan and X_{i+1} is the state of the system in "sampling one out of r" portion of the plan when one or more non-defective items have been sampled. Then $\{X_n , n \geq 1\}$ is a Markov chain.

Sol: In this case,

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases} p, & \text{if } k = 0,\ j = 0,1,2,...i,i+1 \\ 1-p, & \text{if } k = j+1,\ j = 0,1,2,...i \text{ or } k = j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i \\ i+1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & i & i+1 \end{array} \\ \begin{pmatrix} p & 1-p & 0 & \cdots & 0 & 0 \\ p & 0 & 1-p & \cdots & 0 & 0 \\ p & 0 & 0 & 1-p\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p & 0 & 0 & \cdots & 0 & 1-p \\ p & 0 & 0 & \cdots & 0 & 1-p \end{pmatrix} \end{array}$$

Problems

1. A communication system transmits the two digits 0 and 1, each of them passing through several stages. Suppose that the probability that the digit that enters remains unchanged when it leaves is p and that it

changes, is q. Suppose further that X_0 is the digit which enters the first stage of the system and X_n ($n \geq 0$) is the digit leaving the n^{th} stage of the system. Show that $\{X_n\ n \geq 1\}$ forms a Markov chain. Find P, P^2, P^3 and calculate P ($X_2 = 0 \mid X_0 = 0$) and P ($X_3 = 1 \mid X_0 = 1$). Find P^n, and discuss the case when $n \to \infty$.

2. Consider a coin-tossing experiment where the probability of a head is p. Let X_n ($n \geq 0$) is the state of the system at the n^{th} trial denoting the difference of the total number of heads and the total number of tails in n trials. Show that $\{X_n\ n \geq 1\}$ is a Markov chain.

3. Consider two urns containing x balls in each of the two. Out of these $2x$ balls, half are red and half black. At each step of the experiment, one ball is selected at random from each of the urns and the two balls are interchanged. Let X_n is the random variable denoting the number of red balls in the first urn. Show that $\{X_n\ n \geq 1\}$ is a Markov chain.

4. Consider two urns A and B containing a total of N balls. A ball is selected at random from among these N balls and is put in urn A, which has been selected with probability p or the ball is put in urn B, which has been selected with probability q (=1-p). Let X_n is the state of the system represented by the number of balls in urn A. Then show that $\{X_n\ n \geq 1\}$ is a Markov chain.

5. For a t.p.m.

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix};\ 0<a,b<1$$

show that

$$P^n = \frac{1}{a+b}\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$$

Interpret the result when $n \to \infty$.

6. The inventory policy of a company is as follows:

If the supply at the beginning of a time-period is x, then it orders

$$\begin{array}{lll} 0 & \text{if,} & x \geq s \\ s & \text{if,} & x < s \end{array}$$

Assume that there is no loss of time between the demand and the supply. The daily demands are independent with the probability $\{p_k\}$. The demands, which are not met immediately, are lost. Let X_n is the level of inventory at the end of the n^{th} time point. Then $\{X_n\ n \geq 1\}$ is a Markov chain.

7. For a symmetric random walk starting at the origin, find the expected time to return to the origin. If N_n is the number of returns by time-point n, then show that

$$E(N_{2n}) = (2n+1)\binom{2n}{n}\left(\frac{1}{2}\right)^n - 1$$

8. In case of a simple random walk, let 0 be an absorbing barrier and N is a reflecting barrier. Then show that with probability 1, the particle gets absorbed. Find the mean number of steps needed for absorption.