Chapter 1
Stochastic Processes: Some Basic Notions

Key words: Stochastic processes, stationary processes, second-order processes, covariance stationary processes, independent increments, evolutionary processes, Markov processes, Poisson processes, generating functions, probability generating functions, characteristic function, mean, variance, convolutions, compound distributions, Laplace transforms

Suggested readings:

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1.1 Introduction

Stochastic processes are families of random variables, which are functions of, say, time. For example,

1 Consider throwing a die. From this experiment, we can define several stochastic processes.

(i) Let X_n is the outcome of the n^{th} throw, $n \geq 1$.Then $\{X_n, n \geq 1\}$ is a family of random variables such that for a distinct value on n we get a distinct random variable. This process is known as a Bernoulli process.

(ii) Let X_n is the number of aces in the first n throws of a die. For a distinct value on n we get a distinct random variable X_n.

(iii) Let X_n is the maximum number shown in the first n throws. Then $\{X_n, n \geq 1\}$ is a stochastic process. If Y_n is the minimum number shown, then $\{Y_n, n \geq 1\}$ is also a stochastic process. Also, $\{(X_n\ Y_n), n \geq 1\}$ is a stochastic process, which is defined in two dimensions.

2 Consider that there are r cells and an infinitely large number of balls. Let the balls are thrown at random, one by one, into the cells, the ball thrown being equally likely to go into any one of the cells. Suppose that X_n is the number of occupied cells after n throws. Then $\{X_n, \mathrm{n} \geq 1\}$ is a stochastic process. Another stochastic process can be defined by a random variable ${}^{(i)}Y_n$ that denotes the number of balls in i^{th} cell after n throws. Then, for i = 1,2…r, $\{{}^{(i)}Y_k, 1 \leq k \leq n, n \geq 1\}$ is a stochastic process.

3. A model for system reliability can be described with the help of a stochastic process. Consider a system consisting of a number of components, which work simultaneously. When all the components are working, the system is in state 1, and it continues to be in that state for some time. If one or more components of the system fail, the system stops working till it get repaired. A failed system is in state 0. Let X_n is the random variable denoting the state of the system at n^{th} time point. Then $\{X_n, n \geq 1\}$ is a stochastic process.

4. **Brownian motion:** a molecule in a uniform medium (e.g. a liquid or a gas) displays perpetual random movements, which are caused as a result of several forces exerted by the surrounding particles. This motion is called Brownian motion. Let $X(t)$ be the position of the particle at time t, in terms of its distance from its initial position. Obviously $X(0) = 0$. Then $\{X(t), t \in T\}$is a stochastic process, known as Brownian motion process or Wiener process.

5. Consider a random event occurring in time, such as, number of telephone calls received at a switchboard. Suppose that $X(t)$ is the random variable, which represents the number of incoming calls in an interval $(0,t)$ of duration t units. The number of incoming calls in a specified duration is a random variable and $\{X(t), t \in T\}$ constitutes a stochastic process, $T = [0,\infty]$.

1.2 Specification of Stochastic Processes

A stochastic process may be specified by its state space and the time. We now define these two quantities.

(i) **State space of a stochastic process:** is the set of all possible values of an individual random variable X_n.

Discrete state space: If the state space of a stochastic process contains a finite or a denumerable infinity of points then it is called a **discrete state space**, e.g., in example 1(i), {1,2,3,4,5,6} is the state space. The state space in example 1(ii) is given by $\{0,1,2\ldots n, n \geq 1)$. We can write $X_n = Y_1 + Y_2 + \ldots + Y_n$, where Y_i is a discrete random variable denoting the outcome of the i^{th} throw and takes values 0 and 1 according as the i^{th} throw results in an ace or not.

Continuous state space If a state space contains infinitely many points, it is called **continuous state space**. For example, consider $X_n = Z_1 + Z_2 + \ldots + Z_n$, where Z_i is a continuous random variable assuming values in $[0,\infty)$, then the state space of X_n is the interval $[0,\infty)$. Wiener process is also a continuous state space process.

(ii) **Time of a stochastic process:** is the instant over which the stochastic process is defined.

Continuous time stochastic process: are those stochastic processes for which the state of the system is characterized at every instant over a finite or infinite interval.
The number $X(t)$ of incoming calls at a switchboard in an interval $(0,t)$. Here the state space is discrete while the time of the process is continuous. Similar is the case of system reliability model.
If $X(t)$ is the maximum temperature at a particular place in $(0,t)$, then it is a continuous state space stochastic process defined over continuous time. Wiener process is a continuous time, continuous state space process.

Discrete time stochastic process: are those that are defined over distinct time points, e.g., if X_n is the total number of sixes appearing in the first n throws of a die, then the 'time n' implies the throw number n. This is discrete time, discrete state space process.

The process defined by $X_n = Z_1 + Z_2 + \ldots + Z_n$, while Z_t is a continuous random variable in $[0,\infty)$ is a discrete time, continuous state space process.

A discrete time stochastic process is also known as a stochastic sequence. In this case, time is a term or parameter associated with the stochastic processes in general. It may be number, distance, thickness or any other characteristic.

Stochastic processes may be multidimensional also, e.g., $X(t) = (X_1(t), X_2(t))$ where $X_1(t)$ represents the maximum and $X_2(t)$ is the minimum temperature at a place in an interval $(0,t)$ is a two dimensional continuous time, continuous state space process. The process defined in example 1(iii) is a discrete time, discrete state space two-dimensional stochastic process.

Processes with independent increments: A continuous parameter stochastic process $\{X(t), 0 \le t \le \infty\}$ is said to have independent increments, if for all choices of indices $t_0 < t_1 < t_2 < \ldots < t_n$, the random variables $X(t_1) - X(t_0)$, $X(t_2) - X(t_1) \ldots X(t_n) - X(t_{n-1})$ are independent.

In case of discrete parameter process, if $T = \{0,1,2\ldots\}$, $t_i = i\text{-}1$ and $X(t_i) = X_{i-1}$, then $Z_i = X_i - X_{i-1}$, i = 1,2… and $Z_0 = X_0$ are independent.
For a stochastic process with independent increments if the distribution $X(t)$ and $X(t)\text{-}X(s)$, $t,s>0$ are known, then the joint distribution $X(t_1) - X(t_0)$, $X(t_2) - X(t_1), \ldots X(t_n) - X(t_{n-1})$ can be obtained by simple multiplication of probabilities.

Markov process: A stochastic process $\{X(t), 0 \le t \le \infty\}$ is said to be a Markov process if given $X(s)$, the value of $X(t)$, $t>s$ is independent of values of $X(u)$, $u < s$, i.e. if for $t_1 < t_2 < \ldots < t_n < t$, $P(a \le X(t) \le b \mid X(t_1) = x_1, \ldots X(t_n) = x_n) = P(a \le X(t) \le b \mid X(t_n) = x_n)$, then $\{X(t), t \in T\}$ is a Markov process. This physical interpretation of this process is that the future states of this process depend only on the (given) present state, and are independent of the past states.

A discrete parameter Markov process is called a **Markov chain**.

Sample function of a stochastic process: A stochastic process $\{X(t), t \in T\}$ can be viewed as a function of two arguments, $\{X(t, s), t \in T, s \in S\}$, S being the sample space of the random variable X. For a fixed s in S, X (., s)a function of t, is a possible observation on the stochastic process $\{X(t), t \in T\}$and is called a realization or a sample function of the process.

1.3 Stationarity of the stochastic processes:

Stationary processes: A stochastic process $\{X(t), t \in T\}$ is called a stationary process of order n if the joint distribution of random variables $(X(t_1), X(t_2) \ldots X(t_n))$ and $(X(t_1 + h), X(t_2 + h) \ldots X(t_n + h))$ remains same for all $h > 0$.
If this condition holds for any n, then the process is called a strict stationary process.

Stationarity of a process implies that the probabilistic structure of the process is **invariant** under the translation of time-axis.

If mean of a stationary stochastic process exists, then for all real h, $E(X(t)) = E(X(t+h))$

$$\Rightarrow \quad E(X(t)) \quad = \quad \text{m (constant, independent of } t\text{)}$$

If any without any loss of generality, we assume that $m = 0$ and if $C(s,t)$ exists, then

$$\begin{aligned} C(s,t) &= \operatorname{cov}(X(t), X(s)) \\ &= E(X(t)X(s)) - E(X(t))E(X(s)) \\ &= E(X(t)X(s)) \;=\; E(X(t+h)X(s+h)) \quad \forall h>0 \\ &= E(X(t-s)X(0)) \quad for \;\; t \;=\; -s \end{aligned}$$

$$\Rightarrow \quad C(s,t) \;=\; R(|\,t-s\,|)$$

Second-order processes: A real valued, continuous time stochastic process is called a second-order process if $E(X(t))^2 \;<\; \infty$, i e., second order moments are finite.

Mean-value function and covariance kernel of a stochastic process: Let $\{X(t),\; t \in T\}$ be a second-order stochastic process. The mean-value function $m(t)$, for all $t \in T$, is defined as $m(t) \;=\; E(X(t))$ and the covariance kernel $C(s,t)$ is defined as

$$C(s,t) \;=\; \operatorname{cov}(X(s), X(t)) \;=\; E(X(s)X(t)) - E(X(s))E(X(t))$$

A covariance function (also known as auto covariance function) satisfies the following properties:

P1: It is **symmetric** in t and s, i.e. $C(s,t) = C(t,s)$.

P2: **Schwartz inequality** holds, i.e. $C(s,t) \;=\; \sqrt{C(s,s)C(t,t)}$.

P3: It is **non-negative definite**, i.e.

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k C(t_j,t_k) \;=\; E(\sum_{j=1}^{n} a_j X_{t_j})^2 \;\geq\; 0$$

P4: **Closure property:** The sum and the product of two covariance functions are again covariance functions

Then, we can define a **covariance-stationary** process:

A stationary stochastic process $\{X(t),\ t \in T\}$ is called a **covariance-stationary** process if it possesses finite second moments, the mean function is independent of t and the covariance kernel $C(s,t)$ is a function of the absolute difference $| s-t |$ only.

Evolutionary process: A process, which is not stationary, i.e. its distribution doesn't remain stationary as the time progresses, is called an evolutionary process.

Guassian process: If for a stochastic process $\{X(t), t \in T\}$, the distribution of $(X(t_1), X(t_2),\ldots X(t_n))$ $\forall\ t_1, t_2\ldots, t_n \in T$, is multivariate normal , then the process is known as a Guassian process.
If a Guassian process $\{X(t), t \in T\}$ is stationary, then it is strictly stationary.

Martingales (Fair game processes): If for a stochastic process $\{X(t), t \in T\}$,

$$E(X_{t_j} \mid X_{t_{j-1}}, X_{t_{j-2}}, X_{t_{j-2}}, \ldots, X_{t_0}) = X_{t_{j-1}};\ \ t_0 < t_1 < \ldots < t_j \ \in [t_0, \infty) \text{ almost}$$

surely, then $\{X(t), t \in T\}$is called a martingale.

Point process: An integral-valued, continuous time stochastic process $\{X(t), t \in T\}$is called a point or a counting process if the cumulative count can only increase as the time progresses. Poisson process is a point process.

Additive process: Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables taking only the non-negative values. Let

$$S_n = X_1 + X_2 + \ldots + X_n,\quad X_0 = 0$$

Then,

$$S_n = S_{n-1} + X_n$$

$$\Rightarrow\ P(S_n = j \mid S_{n-1} = i, \ldots, S_0 = 0) = P(S_n = j \mid S_{n-1} = i) = P(X_n = j - i)$$

Then $\{S_n, n \geq 1\}$ is called an additive process.

Examples:

(1) Let $X_n, n \geq 1$ be the outcome at the nth throw of a die. Then

$$E(X_n) = \frac{1}{3}(1+2+3+4+5+6) = \frac{8}{3}$$

$$C(n.m) = \text{cov}(n,m) = E(X_n X_m) = 0,\quad m \neq n$$

$$= \frac{65}{18},\quad m=n$$

$$\Rightarrow\ C(n.m) = R(|n-m|)$$

and hence $\{X_n, n \geq 1\}$ is a covariance stationary process.

Since X_n are identically distributed, then $\{X_n, n \geq 1\}$ is strictly stationary.

(2) **Poisson process:** is an evolutionary process.

Consider the process $\{X(t), t \in T\}$with

$$P(X(t)=n) = e^{-at}\frac{(at)^n}{n!}, \quad a>0, \quad n = 0,1,2...$$

$$m(t) = E(X(t)) = at$$

$$C(s,s) = \text{var}(X(s)) = at$$

As mean function and covariance function are functions of t, hence the process is evolutionary.

(3) Consider the process $X(t) = A_1 + A_2 t$, where A_1 and A_2 are independent random variables with $E(A_i) = a_i$ and $\text{Var}(A_i) = \sigma_i^2, \ i = 1,2$. We wish to verify whether the process is stationary or evolutionary.

$$m(t) = E(X(t)) = a_1 + a_2 t$$

$$E(X(t)X(s)) = E((A_1 + A_2 t)(A_1 + A_2 s))$$

$$= E(A_1^2)+(s+t)E(A_1A_2)+stE(A_2^2)$$

$$= (\sigma_1^2+a_1^2)+(s+t)a_1a_2+st(\sigma_2^2+a_2^2)$$

$$\Rightarrow \text{Cov}(X(t)X(s)) = E(X(t)X(s)) - E(X(t))E(X(s))$$

$$= (\sigma_1^2+a_1^2)+(s+t)a_1a_2+st(\sigma_2^2+a_2^2)-(a_1+a_2t)(a_1+a_2s)$$

$$= \sigma_1^2 + st\sigma_2^2$$

which is not independent of t. Hence the process is evolutionary.

(4) Consider the process

$$X(t) = A\cos(wt) + B\sin(wt)$$

where A, B are uncorrelated random variables, each with mean 0 and variance 1 and w is a positive constant. We want to find if the process is stationary or evolutionary.

We have

$$m(t) = \cos(wt)E(A) + \sin(wt)E(B) = 0$$

$$E(X(t)X(s)) = E((A\cos(wt)+B\sin(wt))(A\cos(ws)+B\sin(ws)))$$

$$= E(A^2)\cos(wt)\cos(ws)+E(AB)(\cos(wt)\sin(ws)+\cos(ws)\sin(wt))+E(B^2)\sin(wt)\sin(ws)$$

$$= \cos(w(s-t)) \quad as \quad E(X(t))^2 = \text{var}(X(t)) = 1$$

$$\Rightarrow \; C(s,t) = \cos(w(s-t))$$

Thus first two moments are finite and covariance kernel is a function of difference | s-t | only, hence the process is covariance stationary. The sample function of this process, in fact, represents sine waves of amplitude R and phase θ, where, $R = \sqrt{A^2+B^2}$; $\theta = \tan^{-1}\left(\frac{B}{A}\right)$.

(5) Let X_n for n even, take values +1 and –1, each with probability $\frac{1}{2}$, and for n odd, takes values $+\sqrt{a},\ -\frac{1}{\sqrt{a}}$, with probability $\frac{1}{a+1},\ \frac{a}{a+1}$ respectively. (a is a constant number > -1 and not equal to 0 and 1). Further, X_n let be independent. Show that $\{X_n\}$is covariance but not strictly stationary.

Sol: When n is even

$$P(X_n = 1) = \frac{1}{2} = P(X_n = -1)$$

When n is odd

$$P(X_n = \sqrt{a}) = \frac{1}{a+1}, \quad P(X_n = \frac{-1}{\sqrt{a}}) = \frac{a}{a+1}$$

$$\therefore\; E(X_n) = \sum_n n\ P(X_n = n) = \begin{cases} 1.\frac{1}{2} + (-1).\frac{1}{2} = 0 \text{ when n is even} \\ \sqrt{a}.\frac{1}{a+1} - \frac{1}{\sqrt{a}}.\frac{a}{a+1} = 0 \text{ when n is odd} \end{cases}$$

$$\therefore\; E(X_n) = 0 \quad \forall n.$$

Also, $E(X_n X_m) = 0 \;\; if \;\; n \neq m$ as X_n are independent.

$$E(X_n{}^2) = \sum_n n^2 P(X_n = n)$$

$$= \begin{cases} 1.\frac{1}{2} + 1.\frac{1}{2} = 1, \text{ if } n \text{ is even} \\ a.\frac{1}{a+1} + \frac{1}{a}.\frac{a}{a+1} = 1, \text{ if } n \text{ is odd} \end{cases}$$

$$\Rightarrow \; \text{Var}(X_n) = 1 \;\forall n.$$

Thus, first two moments are finite with 0 mean and constant second order moments. Hence the process is covariance stationary.

Now, consider the random variables $X_{t_1}, ..., X_{t_k}$ where $t_1 = 1,\ t_2 = 3, ...,\ t_k = k^{th}$ odd number and $h = 1. \Rightarrow\ t_1 + h = 2, ...,\ t_k + h = k^{th}$ even number. Then $X_{t_1}, ..., X_{t_k}$ and $X_{t_1+h}, ..., X_{t_k+h}$ are not identically distributed. Hence the process is not strictly stationary.

(6) Let $Y_n = a_0 X_n + a_1 X_{n-1}$ $(n = 1, 2, ...)$ where a_0, a_1 are constants and X_n $(n = 0,1,2,\ldots)$ are i.i.d. random variables with mean 0 and variance σ^2. Is $\{Y_n,\ n \geq 1\}$ covariance stationary? Is it a Markov process?

Sol:

$$E(Y_n) = a_0 E(X_n) + a_1 E(X_{n-1}) = 0$$

$$E(Y_n Y_m) = E((a_0 X_n + a_1 X_{n-1})(a_0 X_m + a_1 X_{m-1}))$$

$$= a_0^2 E(X_n X_m) + a_0 a_1 E(X_m X_{n-1} + X_n X_{m-1}) + a_1^2 E(X_{n-1} X_{m-1})$$

$$= \begin{cases} a_0^2 + a_1^2, & \text{if } n = m \\ a_0 a_1, & \text{if } n = m-1 \text{ or } m = n-1 \\ 0, & \text{otherwise} \end{cases}$$

Since the second order moments are finite, mean value function is independent of *t* and covariance kernel is a function of | *n-m* | only, hence the process is covariance stationary.

Now,

$$E(Y_{n+1}^2 \mid y_n, ..., y_0) = E(a_0^2 X_{n+1}^2 + a_1^2 X_n^2 + 2a_0 a_1 X_n X_{n+1} \mid y_n, ..., y_0)$$

$$= (a_0^2 + a_1^2)\sigma^2$$

$$= E(Y_{n+1}^2)$$

Hence the process is not Markov.

(7) Let $X(t) = \sum_{r=1}^{k} (A_r \cos\theta_r t + B_r \sin\theta_r t)$ where A_r, B_r are uncorrelated random variables with mean 0, variance σ^2 and θr are constants. Show that { $X(t)$ }is covariance stationary.

Sol:

$$E(X(t)) = \sum_{r=1}^{k} (E(A_r)\cos\theta_r t + E(B_r)\sin\theta_r t) = 0$$

$$E(X(t), X(s)) = E\left(\sum_{r=1}^{k} (A_r \cos\theta_r t + B_r \sin\theta_r t) \sum_{r=1}^{k} (A_r \cos\theta_r s + B_r \sin\theta_r s)\right)$$

$$= \sum_{r=1}^{k} (E(A_r^2)\cos\theta_r t\cos\theta_r s + \sum_{r=1}^{k} E(B_r^2)\sin\theta_r t\sin\theta_r s$$

as A_r and, B_r are uncorrelated

$$= \sum_{r=1}^{k} (\cos\theta_r t\cos\theta_r s + \sin\theta_r t\sin\theta_r s)\sigma^2$$

$$= \sum_{r=1}^{k} (\cos(t-s)\theta_r)\sigma^2$$

which depends on | t-s | only. Hence { $X(t)$ }is covariance stationary.

(8) Let $X(t) = A_0 + A_1 t + A_2 t^2$, where A_i, i=0,1,2 are uncorrelated random variables with mean 0 and variance 1.Find the mean value function and covariance function of {X(t)}.

Sol:

$$m(t) = E(X(t)) = E(A_0 + A_1 t + A_2 t^2) = 0$$

$$Cov(X(t), X(s)) = E(X(t)X(s))$$

$$= E((A_0+A_1 t+A_2 t^2)(A_0+A_1 s+A_2 s^2))$$

$$= E(A_0^2)+stE(A_1^2)+t^2E(A_2^2)$$

$$= 1+st+t^2$$

and $\text{Var}(X(t)) = 1+t^2+t^4$

(9.) Find the covariance function of $\{Y_n\}$ given by

$$Y_n = a_0 X_n + a_1 X_{n-1} + \ldots + a_k X_{n-k}, \quad n = 1, 2, \ldots$$

where a_i's are constants and X_n's are uncorrelated random variables.

Sol:

$$E(Y_n) = \sum_{r=0}^{k} a_{n-r} E(X_r)$$

$$E(Y_n Y_m) = E\left(\left(\sum_{r=0}^{k} a_{n-r} E(X_r)\right)\left(\sum_{r=0}^{k} a_{m-r} E(X_r)\right)\right)$$

$$= E((a_0 X_n + a_1 X_{n-1} + \ldots a_k X_{n-k})(a_0 X_m + a_1 X_{m-1} + \ldots a_k X_{m-k}))$$

Since X_i's are uncorrelated so

$$\operatorname{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j) = 0$$

$$\Rightarrow E(X_i X_j) = E(X_i)E(X_j), \; i \neq j$$

Without any loss of generality, let

$$\left.\begin{array}{l} E(X_i) = 0 \\ \operatorname{Var}(X_i) = 1 \end{array}\right\} \forall i$$

$$\Rightarrow E(Y_n Y_m) = \begin{cases} a_0{}^2 + a_1{}^2 + \ldots + a_k{}^2, & \text{if } n = m \\ \sum_{r=0}^{k-1+i} a_r a_{r+1-i} & \text{if } n = m - i \text{ or } m = n - i \\ 0, & \text{otherwise} \end{cases}$$

$$\text{and } E(Y_n) = 0$$

$$\Rightarrow \operatorname{cov}(Y_n, Y_m) = \begin{cases} a_0{}^2 + a_1{}^2 + \ldots + a_k{}^2, & \text{if } n = m \\ \sum_{r=0}^{k-1+i} a_r a_{r+1-i} & \text{if } n = m - i \text{ or } m = n - i \\ 0, & \text{otherwise} \end{cases}$$

1.4 Tools and Techniques of Stochastic Processes.

Although several statistical techniques can be used in dealing with the stochastic processes, yet generating functions provide a powerful tool for dealing with the same. The principal advantage of this technique is that a single function may be used to represent a whole set of individual terms.

Definition (Generating Functions): Consider a sequence a_0, a_1, …a_n,… of real numbers. Define a function

$$A(s) = a_0 + a_1 s + a_2 s^2 + \ldots = \sum_{n=0}^{\infty} a_n s^n \tag{1.1}$$

If this power series converges in some interval $-s_0 < s < s_0$, then A(s) is called generating function of the sequence $\{a_n\}$ and the interval $-s_0 < s < s$ is called the radius of convergence.
Within the radius of convergence,

$$\frac{d}{ds} \sum_{n=0}^{\infty} a_n s^n = \sum_{n=0}^{\infty} \frac{d}{ds} a_n s^n \tag{1.2}$$

If A(s) is differentiated *k* times at *s* = 0, i.e.

$$\frac{d^k}{ds^k} A(S)\Big|_{s=0} = k!\,a_k$$
$$\Rightarrow \quad a_k = \frac{1}{k!}\frac{d^k}{ds^k} A(S)\Big|_{s=0} \tag{1.3}$$

Probability Generating Function: If X is a non-negative integral-valued random variable, assuming non-zero probabilities, i.e.

$$\mathrm{P}(X = k) = p_k\,,\; k = 0,1,2,...;\; \sum_k p_k = 1$$

Then, the series defined by

$$P(s) = p_0 + p_1 s + p_2 s^2 + ... = \sum_{n=0}^{\infty} p_n s^n \tag{1.4}$$

is called the probability generating function of the random variable X. The series is necessarily convergent and infinitely differentiable. Also,

$$P(1) = 1$$

The p.g.f. P(s) determines $\{p_k\}$ uniquely. Also,

$$P(s) = \sum_{k=0}^{\infty} \Pr\{X = k\} s^k = E(s^X) \tag{1.5}$$

We have the following results:

R1:
$$P'(s) = p_1 + 2p_2 s + 3p_3 s^2 + ... = \sum_{k=1}^{\infty} kp_k s^{k-1}$$
$$\Rightarrow \; P'(1) = \sum_{k=0}^{\infty} kp_k = \sum_{k=0}^{\infty} kP\{X = k\} = E(X) \tag{1.6}$$

R2:

$$P''(s) = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-2}$$
$$\Rightarrow P''(1) = \sum_{k=2}^{\infty} k(k-1)P(X = k) = E(X(X-1))$$

Hence,
$$\mathrm{Var}(X) = E(X^2) - E^2(X) = E(X(X-1)) + E(X) - E^2(X)$$
$$= P''(1) + P'(1) - [P'(1)]^2 \tag{1.7}$$

R3: The k^{th} factorial moment is given by

$$E\left(X(X-1)...(X-k+1)\right) = \frac{d^k}{ds^k} P(s)\Big|_{s=1},\; k = 1,2,... \tag{1.8}$$

R4:

$$\int_0^1 P(s)ds = \int_0^1 \sum_{k=0}^{\infty} p_k s^k ds = \sum_k p_k \int_0^1 s^k ds$$

$$= \sum_k \frac{1}{k+1} p_k = E\left(\frac{1}{1+X}\right)$$

R5: Let $q_k = P(X > k) = p_{k+1} + p_{k+2} + ..., \quad k = 0,1,2,...$

$$\Rightarrow q_k - q_{k+1} = p_{k+1}$$

$$\Rightarrow \sum_{k=0}^{\infty} q_k s^k - \frac{1}{s}\sum_{k=0}^{\infty} q_{k+1} s^{k+1} = \frac{1}{s}\sum_{k=0}^{\infty} p_{k+1} s^{k+1}$$

$$\Rightarrow Q(s) - \frac{1}{s}[Q(S) - q_0] = \frac{1}{s}[P(s) - p_0]$$

and, $q_0 = p_1 + p_2 + ... = 1 - p_0$

$$\Rightarrow Q(s) = \frac{1-P(s)}{1-s} \tag{1.9}$$

R6: Let $r_k = \frac{P(X>k-1)}{k}, \quad k = 1,2,...$

$$\Rightarrow kr_k = P(X > k-1) = q_{k-1}$$

$$\Rightarrow \sum_{k=1}^{\infty} kr_k s^{k-1} = \sum_{k=1}^{\infty} q_{k-1} s^{k-1}$$

$$\Rightarrow Q(s) = \frac{d}{ds}\sum_{k=1}^{\infty} r_k s^k = \frac{d}{ds} R(s)$$

or, $R(s) = \int_0^s Q(y)dy$, where R(s) is the generating function of $\{r_k\}$.

R7: Let X be a random variable with p.g.f. P(s). Let $Y = mX + n; \; m,n \in \mathrm{I}, \; m \neq 0$

$$\Rightarrow P_Y(s) = P(s^Y) = P(s^{mX+n}) = s^n P(s^{mX})$$

$$= s^n E(s^m)^X = s^n P_X(s^m)$$

Characteristic functions: Let X be a random variable. Then its characteristic function $\phi_x(.)$ is defined as

$\phi_x(s) = \mathrm{E}\left(e^{isX}\right); \; i = \sqrt{-1}$, where s is a real number.

A random variable **always** possesses a characteristic function, even though it may not possess a finite moment or moment generating function. Also a characteristic function represents a random variable uniquely. There is a one-to-one correspondence between the distribution function and the characteristic function of a random variable.

For jointly distributed random variables X_1, $X_2 \ldots X_n$, the joint characteristic function $\phi_{X_1,X_2,\ldots X_n}(.,.,\ldots,.)$ is defined as

$$\phi_{X_1,X_2,\ldots X_n}(s_1,s_2,\ldots,s_n) = E\left(e^{i(s_1X_1+s_2X_2+\ldots+s_nX_n)}\right)$$

$$= E\left(e^{i(s_1X_1)}\ e^{i(s_2X_2)}\ldots e^{i(s_nX_n)}\right)$$

$$= E\left(\prod_{j=1}^{n} e^{i(s_jX_j)}\right)$$

If the jointly distributed random variables X_1, $X_2 \ldots X_n$ are independent also, then

$$\phi_{X_1,X_2,\ldots X_n}(s_1,s_2,\ldots,s_n) = E\left(\prod_{j=1}^{n} e^{i(s_jX_j)}\right)$$

$$= \prod_{j=1}^{n} e^{i(s_jX_j)}$$

$$= \prod_{j=1}^{n} \phi_{X_j}(s_j)e^{i(s_jX_j)}$$

1.5 Determination of probabilities using characteristic functions:

For a random variable X,

$$P(X = x) = \lim_{U\to\infty} \frac{1}{2U}\int_{-U}^{U} \phi_X(s)\,ds;\ -\infty < x < \infty$$

If X is a discrete random variable, then

$$p_k = P(X = k) = \lim_{U\to\infty} \frac{1}{2U}\sum_{-U}^{U} \phi_X(s);\ -\infty < x < \infty$$

This representation is unique.

If the moments of the random variable X exist, they can be determined using characteristic functions as follows:

$$E(X^r) = \frac{1}{\iota^r}\frac{\partial^r}{\partial s^r}\phi_X(s)\Big|_{s=0}\ ;\ r = 1, 2, 3\ldots$$

For jointly distributed random variables, X_1, $X_2 \ldots X_n$,

$$E(X_1^{r_1} X_2^{r_2} \ldots X_k^{r_k}) = \frac{1}{\iota^k} \frac{\partial^k}{\partial s_1^{r_1} \partial s_2^{r_2} \ldots \partial s_k^{r_k}} \phi_{X_1 X_2 \ldots X_n}(s, s \ldots s)\Big|_{s=0}$$

$$k = r_1 + r_2 + \ldots + r_k$$

Representing a stochastic process in terms of joint characteristic function of the random variables:

We have seen that a stochastic process $\{X(t),\ t \in T\}$can be identified and represented in terms of the (joint) distribution of a finite number of members of the family although the index set T may be infinite. Then, there being a one-to-one correspondence between the distribution function and the characteristic function, a stochastic process may be represented in terms of (joint) characteristic function of the member(s) of the family. For example, consider a sequence of independent random variables X_1, X_2 …X_n, and the sequence $\{S_n = X_1 + X_2 + \ldots + X_n,\ n \geq 1\}$ of the partial sums. Then

$$\phi_{S_1, S_2, \ldots S_n}(s_1, s_2, \ldots, s_n) = \phi_{X_1}(s_1 + s_2 + \ldots + s_n)\phi_{X_2}(s_2 + \ldots + s_n) \ldots \phi_{X_n}(s_n)$$

as $\sum_{k=1}^{n} s_k S_k = s_n(S_n - S_{n-1}) + (s_{n-1} + s_n)(S_{n-1} - S_{n-2}) + \ldots + (s_2 + s_3 + \ldots + s_n)\ (S_2 - S_1) + (s_1 + s_2 + \ldots + s_n)S_1$

For a stochastic process with independent increments,

$$\phi_{X(t_1), X(t_2), \ldots X(t_n)}(s_1, s_2, \ldots, s_n) = \phi_{X(t_1)}(s_1 + s_2 + \ldots + s_n) \prod_{k=2}^{n} \phi_{(X(t_k) - X(t_{k-1}))}(s_k + \ldots + s_n)$$

In this case, if the distribution of $X(t_1)$ and $X(t) - X(s)$ are known for all t and s > 0, then the joint distribution of $X(t_1)$, $X(t_2)$… $X(t_n)$ can be determined.

1.6 Examples

(10) Consider the sequence $\left\{a_k = \frac{1}{k!}\right\}$ of real numbers. The generating function of this sequence is given by

$$A(s) = \sum_{k=0}^{\infty} \frac{1}{k!} s^k = e^s$$

(11) Let X ~ U(a,b)

$$\Rightarrow \; P(X=k) = \frac{1}{b-a}; \quad k \in (a,b)$$

$$\Rightarrow \; P(s) = \sum_{k=a}^{b} \frac{1}{b-a} s^k = \frac{1}{b-a} s^a \sum_{k=0}^{b-a} s^k$$

$$= \frac{s^a}{b-a}\left(\frac{1-s^{b-a+1}}{1-s}\right)$$

$$= \left(\frac{s^a - s^{b-a+1}}{(1-s)(b-a)}\right)$$

In this case,

$$\phi_X(s) = \left(\frac{e^{isb}-e^{isa}}{is(b-a)}\right)$$

If a = 0 and b = 1, then

$$\phi_X(s) = \left(\frac{e^{is}-1}{is}\right)$$

$$P(s) = \frac{1-s^2}{1-s} = 1+s$$

$$P'(1) = 1 = E(X)$$

$$P''(1) = 0 \Rightarrow \mathrm{Var}(X) = 0$$

(12) **Bernoulli distribution:** Let X ~ Ber(1,p) such that

$$P(X=1) = p; \; P(X=0) = q = 1-p; \; 0 < p < 1$$

$$\Rightarrow \; P(s) = E(s^X) = q + ps$$

$$E(X) = P'(1)$$

$$\mathrm{Var}(X) = P'(1) - [P'(1)]^2 = p - p^2 = pq$$

$$\phi_X(s) = \left(pe^{is}+q\right)$$

(13) **Binomial distribution:** Let X ~ Bin(n,p) such that

$$\Rightarrow \ p_k = P(X=k) = \binom{n}{k} p^k q^{n-k},$$

$$k = 0,1,2\ldots n; \quad p,q > 0; \quad p+q = 1$$

$$\Rightarrow \ P(s) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=0}^{\infty} \binom{n}{k} p^k q^{n-k} s^k$$

$$= (q+ps)^n$$

$$E(X) = P'(1) = np$$

$$\mathrm{Var}(X) = P''(1) + P'(1) - [P'(1)]^2$$

$$= n(n-1)p^2 + np - n^2 p^2$$

$$= np - np^2$$

$$= npq$$

$$\phi_X(s) = \left(pe^{is} + q\right)^n$$

(14) **Poisson distribution:** Let X~ Pois(λ)

$$\Rightarrow \ p_k = P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}; \ k = 0,1,2,\ldots; \ \lambda > 0$$

$$\Rightarrow \ P(s) = \sum_{k=0}^{\infty} p_k s^k = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k s^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

$$E(X) = P'(1) = \lambda$$

$$P''(1) = \lambda^2 \ \Rightarrow \ \mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\phi_X(s) = e^{\lambda}\left(e^{is} - 1\right)$$

(15) **Geometric distribution**: Let X~ Geo(p)

$$\Rightarrow \quad p_k \;=\; P\{X = k\} \;=\; q^k p;$$

$$k = 0,1,2,\ldots; \quad p,q > 0; \quad q = 1-p$$

$$\Rightarrow P(s) = \sum_{k=0}^{\infty} q_k\, p s^k = \frac{p}{1-qs}$$

$$E(X) = P'(1) = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

$$P''(1) = \frac{2pq^2}{(1-q)^3} = \frac{2q^2}{p^2} \Rightarrow \operatorname{Var}(X) = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q}{p^2}$$

$$\phi_X(s) = \frac{pe^{is}}{1-qe^{is}}$$

Non-ageing property of Geometric Distribution: For a geometric variable X,

$$P(X = s + r \mid X \geq s) = \frac{P(X=s+r, X \geq s)}{P(X \geq s)}$$

$$= \frac{P(X=s+r)}{\sum_{k=0}^{\infty} q^k p} = \frac{q^{s+r}p}{\frac{q^s p}{1-q}} = q^r p$$

$$= P(X=r)$$

$$\Rightarrow P(X = s + r \mid X \geq s) = P(X = r)$$

This property, called memoryless or non-ageing property, is a characterization of geometric distribution among all discrete non-negative integral-valued distributions.

(16) **Logarithmic series distribution:** Let X has a logarithmic series distribution. Then

$$p_k = P(X=k) = \frac{\alpha q^k}{k}; \quad k = 1, 2, 3...$$

$$\alpha = \frac{-1}{\log p}, \quad 0 < p, q < 1$$

$$\Rightarrow \; P(s) = \sum_{k=1}^{\infty} p_k s^k = \sum_{k=1}^{\infty} \frac{\alpha q^k s^k}{k} = -\alpha \log(1-qs)$$

$$= \frac{\log(1-qs)}{\log(1-q)}$$

$$\Rightarrow \; E(X) = \frac{-q}{(1-sq)\log(1-q)}\Big|_{s=1} = \frac{\alpha q}{1-q}$$

$$P''(1) = \frac{q^2}{(1-sq)^2 \log(1-q)}\Big|_{s=1} = \frac{\alpha q^2}{(1-q)^2}$$

$$\Rightarrow \; \text{Var}(X) = \frac{\alpha q^2}{(1-q)^2} + \frac{\alpha q}{1-q} - \frac{\alpha^2 q^2}{(1-q)^2}$$

$$= \frac{\alpha q(1-\alpha q)}{(1-q)^2}$$

$$\phi_X(s) = \frac{\log(1-e^{is})}{\log(1-q)}$$

(17) Find the generating function of the Fibonacci numbers $\{ f_n \}$ defined by

$$f_n = f_{n-1} + f_{n-2}; \quad n \geq 2; \quad f_0 = 0, \quad f_1 = 1$$

Sol: The gf of the series is given by

$$F(s) = \sum_{n=0}^{\infty} f_n s^n = f_0 + f_1 s + \sum_{n=2}^{\infty} f_n s^n$$

$$= s + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) s^n$$

$$= s + sF(s) + s^2 F(s)$$

$$\Rightarrow (1-s-s^2)F(s) = s$$

$$\Rightarrow F(s) = \frac{s}{1-s-s^2}$$

(18) Let X be a random variable denoting the number of tosses required to get two consecutive heads when a fair coin is tossed. Show that the p.g.f. of X is

$$\left(\frac{s^2}{4}\right)\left(1 - \frac{s}{2} - \left(\frac{s}{2}\right)^2\right)^{-1}$$

Sol: The event can occur in the following ways:

$$HH,\ THH,\ TTHH,\ HTHH, \ldots$$

Let $P(X = k) = p_k$

$$\Rightarrow P(X=0) = f_0 = f_1 = P(X=1)$$

$$P(X = 2) = f_2 = P(HH) = p^2$$

$$P(X = 3) = f_3 = P(THH) = qp^2 = qf_2 + qpf_1$$

$$P(X = 4) = f_4 = P(TTHH, HTHH) = q^2p^2 + qp^3 = qf_3 + qpf_2$$

$$P(X = 5) = f_5 = P(TTTHH, THTHH, HTTHH) = q^3p^2 + 2q^2p^3 = qf_4 + qpf_3$$

So, in general,

$$f_n = P(X = n) = qf_{n-1} + qpf_{n-2}$$

$$= \frac{1}{2} f_{n-1} + \frac{1}{4} f_{n-2}$$

since the coin is a fair coin.

$$\Rightarrow F(s) = \sum_{n=0}^{\infty} f_n s^n = f_2 s^2 + \sum_{n=3}^{\infty}\left(\frac{1}{2} f_{n-1} + \frac{1}{4} f_{n-2}\right) s^n$$

$$= \frac{s^2}{4} + \frac{s}{2} F(s) + \frac{s^2}{4} F(s)$$

$$\Rightarrow F(s) = \frac{\frac{s^2}{4}}{1 - \frac{s}{2} - \frac{s^2}{4}}$$

1.7 Determination of $\{p_k\}$ from $P(s)$

There are two methods, which can be used to determine the sequence of probabilities $\{p_k\}$.

(i) If $P(s) = \sum_{k=0}^{\infty} p_k s^k$ is exactly differentiable, then

$$\frac{d^k(P(s))}{ds^k}\Big|_{s=0} = k!\, p_k \Rightarrow p_k = \frac{1}{k!} \frac{d^k(P(s))}{ds^k}\Big|_{s=0}$$

This representation is unique.

This is a direct and simple method of extracting probabilities from a probability generating functions, provided the functions are easily differentiable. But this may not be the case always and I may not be possible to obtain probabilities in explicit terms. Then a sophisticated and elegant method of obtaining probabilities is partial fraction theorem.

(ii) **Partial Fraction Theorem:** If $P(s)$ is a rational p.g.f. of the form $\frac{U(s)}{V(s)}$, where U and V are polynomials without any common roots, then

$$P(s) = \frac{a_1}{s_1-s}+...+\frac{a_r}{s_r-s}$$

is the partial fraction decomposition of $P(s)$ where $a_i = -\frac{U(s_i)}{V'(s_i)}$. Then

$$p_k = \sum_{i=1}^{r}\frac{a_i}{s_i^{k+1}}$$

1.8 Convolutions: Sum of a fixed number of random variables

Let X and Y are two independent, non-negative integral-valued random variables with the probability distributions $P(X = k) = a_k$; $P(Y = j) = b_j$ respectively. Define $Z = X+Y$. Then Z is also a random variable with the probability distribution

$$c_r = P(Z = r) = \sum_{k=0}^{r} P(X = k, Y = r-k)$$

$$= \sum_{k=0}^{r} a_k b_{r-k}\text{, as } X \text{ and } Y \text{ are independent.}$$

The sequence $\{c_r\}$ of probabilities, which is a combination of sequences $\{a_r\}$ and $\{b_r\}$ and is denoted by $\{c_r\}=\{a_r\}*\{b_r\}$ is called convolution of $\{a_r\}$ and $\{b_r\}$.

We state and prove the following results.

Theorem1.1: The p.g.f. of the sum of two independent random variables X and Y is the product of p.g.f.'s of X and Y.

Proof: Let $A(s)$, $B(s)$ and $C(s)$ are the p.g.f.'s of X, Y and Z respectively. Then

$$C(s) = E(s^{X+Y}) = E(s^X)E(s^Y) = A(s)B(s)$$

Alternatively, if $\{c_j\}=\{a_j\}*\{b_j\}$, then

$$C(s) = \sum_{j=0}^{\infty} c_j s^j = \sum_{j=0}^{\infty} s^j \sum_{k=0}^{j} a_k b_{j-k}$$

$$= \sum_{k=0}^{\infty} a_k s^k \sum_{j=k}^{\infty} b_{j-k} s^{j-k} = \sum_{k=0}^{\infty} a_k s^k \sum_{j=0}^{\infty} b_j s^j = A(S)B(s)$$

Theorem1.2: The sum $S_n = X_1 + X_2 + ... + X_n$ of a fixed number n of i.i.d. random variables X_i has the p.g.f. $[P(s)]^n$, each X_i having the p.g.f. $P(s)$.

Proof: The result can be proved easily by induction since for $n = 2$, the p.g.f. of S_n is

$$P_{S_2}(s) = P(s).P(s) = [P(s)]^2$$

and, extending the result to a general n, we have

$$P_{S_n}(s) = [P(s)]^n$$

1.9 Examples

(19) **Sum of independent Bernoulli variables:**

If X_i's are i.i.d. Ber(p) variables, then

$$P(s) = q + ps$$

$$\Rightarrow P_{S_n}(s) = (q+ps)^n$$

$$\Rightarrow S_n \sim Bin(n, p)$$

(20) **Sum of independent Poisson variables:**

Let $X_i \sim \text{Pois}(\lambda_i),\ i = 1,2$ and $Z = X_1 + X_2$. Then

$$P_{X_i}(s) = e^{\lambda_i(s-1)},\quad i = 1,2$$

$$\Rightarrow P_Z(s) = P_{X_1}(s)P_{X_2}(s)$$

$$= e^{\lambda_1(s-1)} e^{\lambda_2(s-1)} = e^{(\lambda_1+\lambda_2)(s-1)}$$

i.e. $Z \sim \text{Pois}(\lambda_1 + \lambda_2)$.

In general, if $X_i \sim \text{Pois}(\lambda_i),\ i = 1,2,...n$ and $S_n = X_1 + X_2 + ... + X_n$, then $S_n \sim \text{Pois}(\lambda_1 + \lambda_2 + ... + \lambda_n)$.

(21). **Sum of independent geometric variables:**

Let $X_i \sim \text{Geo}(p),\ i = 1, 2$ i.i,d, variables i.e.

$$P(X_i = k) = q^k p \text{ and } P_{X_i}(s) = \frac{p}{1-qs}.$$

If $Z = X_1 + X_2$, then

$$P_Z(s) = \left(\frac{p}{1-qs}\right)^2 = p^2(1-qs)^{-2} = p^2 \sum_{r=0}^{\infty} \binom{r+1}{r}$$

$$\Rightarrow P(Z = r) = \text{coefficient of } s^r \text{ in } P_Z(s)$$

$$= \binom{r+1}{r} p^2 q^r = (r+1) p^2 q^r, \quad r \geq 0.$$

If X_1 denote the number of failures preceding the first success and X_2 be the number of failures following the first success and preceding the second success, then $Z = X_1 + X_2$ is the number of trials preceding the second success. i.e.

$$\underbrace{\underbrace{FF...FS}_{X_1}\ \underbrace{F...FFS}_{X_2}}_{X_1+X_2}$$

and $Z \sim NB(2,p)$.

In general, if $S_n = X_1 + X_2 + ... + X_n$ is the sum of n i.i.d. geometric variables, then

$$P_{S_n}(s) = (P(s))^n = \frac{p^n}{(1-qs)^n}$$

$$= p^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} q^k s^k$$

$$\Rightarrow P(S_n = k) = \binom{n+k-1}{k} q^k p^n$$

is the distribution of trials preceding the n^{th} success i.e. $S_n \sim NB(n, p)$

(22) If X, Y are independent random variables with p.g.f. A(s) and B(s) respectively, then

$$D(s) = \text{pgf of} \{Z = X - Y\} = E\left(s^X \left(s^{-1}\right)^Y\right) = E(s^X)E\left(\frac{1}{s}\right)^Y = A(s)B(s)$$

1.10 Sum of a random number of discrete random variables:

Sometimes, we come across situations when the sum of a random number N of random variables is to be considered. For example, if X_i denotes the number of persons involved in the i^{th} accident in a day in a certain city and if the number of accidents N happening in a particular day is a random variable, which is independent of Xi's, then the sum $S_N = X_1 + X_2 + ... + X_N$ is the total number of persons involved in accidents on a particular day.

Now, we obtain the p.g.f. of S_N:

Theorem 1.3: Let X_i, $i = 1,2\ldots$ be i.i.d. random variables with $P(X_i = k) = p_k$ and p.g.f. $P(s) = \sum_k p_k s^k$, $i = 1, 2, ...$ Define $S_N = X_1 + X_2 + ... + X_N$, where N is a random variable independent of X_i's. Let the distribution function of N is given by $P(N = n) = g_n$ and p.g.f. $G(s) = \sum_n g_n s^n$ .Then, the p.g.f. $H(s)$ of S_N is given by the compound function $G(P(s))$, i.e. $H(s) = \sum_j P(S_N = j)s^j = G(P(s))$.

Proof: $H(s) = \sum_{j=0}^{\infty} h_j s^j = \sum_{n=0}^{\infty} P(S_n = j)P(N = n)$, where $\{h_j\}$ is the probability distribution of S_N, so

$$h_j = P(S_N = j) = \sum_{n=0}^{\infty} P(S_N = j \mid N = n)P(N = n)$$

$$= \sum_{n=0}^{\infty} P(S_n = j)P(N = n) \text{ as the events are independent}$$

$$= \sum_{n=0}^{\infty} P(S_n = j)g_n = \sum_{n=0}^{\infty} (p_j)^n s^n$$

$$\Rightarrow H(s) = \sum_j h_j s^j = \sum_j \sum_{n=0}^{\infty} (p_j)^n g_n s^j$$

$$= \sum_{n=0}^{\infty} g_n \sum_j (p_j)^n s^j = \sum_{n=0}^{\infty} g_n (P(s))^n = G(P(s))$$

Mean and variance of S_N:

$$E(S_N) = H'(s)|_{s=1} = P'(s)G''(P(s))|_{s=1}$$

$$= P'(1)G'(1) = E(X)E(N)$$

Again, $H''(s) = (P'(s))^2 G'(P(s)) + P''(s)G'(P(s))|_{s=1}$

and, $\text{Var}(S_N) = E(S_N{}^2) - E^2(S_N)$

where, $E(S_N{}^2) = H''(s)|_{s=1} + H'(s)|_{s=1}$

$$= P''(1)G'(1) + G''(1)(P'(1))^2 + E(N)E(X)$$

$$= P''(1)E(N) + G''(1)(E(X))^2 + E(N)E(X)$$

$$= (E(X^2) - E(X))E(N) + (E(N^2) - E(N))E(X) + E(N)E(X)$$

$$= E(X^2)E(N) - E(N)E^2(X) + E(N^2)E^2(X)$$

$$= E(N)\text{Var}(X) + E(N^2)E^2(X)$$

$$\Rightarrow \text{Var}(S_N) = E(N)\text{Var}(X) + (E(N^2) - E^2(N))E^2(X)$$

$$= E(N)\text{Var}(X) + E^2(X)\text{Var}(N)$$

Alternatively,

$$P(S_N = j) = \sum_{n=0}^{\infty} P(S_N = j | N = n)P(N = n)$$

$$= \sum_{n=0}^{\infty} P(N = n)P(S_n = j)$$

$$E(S_N) = \sum_{j=0}^{\infty} jP(S_N = j)$$

$$= \sum_j j \sum_{n=0}^{\infty} P(N = n)P(S_n = j)$$

$$= \sum_n P(N = n)\sum_j jP(S_n = j)$$

$$= \sum_n P(N = n)E(S_n)$$

where, $E(S_n) = E(X_1 + X_2 + ... + X_n)$

$$\Rightarrow E(S_N) = \sum_n P(N = n).nE(X)$$

$$= E(X)\sum_n nP(N = n) = E(X)E(N)$$

$$\text{Var}(S_N) = E(S_N{}^2) - E^2(S_N)$$

where, $E(S_N{}^2) = \sum_j j^2 P(S_N = j)$

$$= \sum_j j^2 \sum_n P(N = n) P(S_n = j)$$

$$= \sum_n P(N = n) \sum_j j^2 P(S_n = j)$$

$$= \sum_n P(N = n) E(S_n{}^2)$$

where, $$E(S_n{}^2) = E(X_1 + X_2 + \ldots + X_n)^2$$

$$= E(X_1 + X_2 + \ldots + X_n)^2$$

$$= E(nX_1{}^2 + n(n-1)X_1X_2)$$

$$= nE(X^2) + n(n-1)E^2(X)$$

since X_i's are i.i.d.

$$\Rightarrow E(S_N{}^2) = \sum_n P(N = n)(nE(X^2) + n(n-1)E^2(X))$$

$$= E(N)E(X^2) + (E(N^2) - E(N))E^2(X)$$

$$\Rightarrow \text{Var}(S_N{}^2) = E(N)E(X^2) + E(N^2)E^2(X) - E(N)E^2(X) - E^2(N)E^2(X)$$

$$= E(N)(E(X^2) - E^2(X)) + E(X^2)(E(N^2) - E^2(N))$$

$$= E(N)\,\text{var}(X) + E^2(X)\,\text{var}(N)$$

Deduction: $$\frac{Var(S_N)}{E(S_N)} = \frac{Var(X)}{E(X)} + E(X)\frac{Var(N)}{E(X)}$$

since, $E(S_N) = E(N)E(X)$

Examples:

(23) **Compound Poisson distribution**: Let $N \sim \text{Pois}(\lambda)$, $\lambda > 0$.

Then, $$G(s) = \sum_n g_n s^n = e^{\lambda(s-1)}$$

and $$H(s) = G(P(s)) = e^{\lambda(P(s)-1)}$$

H(s) is p.g.f. of compound Poisson distribution

$$E(S_N) = \lambda e^{\lambda(P(s)-1)} P'(s)\,|_{s=1}$$

$$= \lambda P'(1) = \lambda E(X)$$

$$\text{Var}(S_N) = (H''(s) + H'(s) - (H'(s))^2)|_{s=1}$$

$$= ((\lambda P'(s))^2 e^{\lambda(P(s)-1)} + \lambda e^{\lambda(P(s)-1)} P''(s))|_{s=1} + \lambda E(X) - \lambda^2 E^2(X)$$

$$= \lambda^2 E^2(X) + \lambda P''(1) + \lambda E(X) - \lambda^2 E^2(X)$$

$$= \lambda P''(1) + \lambda E(X)$$

$$= \lambda\left(\text{Var}(X) + E^2(X)\right)$$

$$\Rightarrow \frac{\text{Var}(S_N)}{E^2(S_N)} = \frac{E(N)\text{Var}(X) + \text{Var}(N)E^2(X)}{E^2(X)E^2(N)}$$

$$= \frac{\lambda \text{Var}(X) + \lambda E^2(X)}{\lambda E^2(X)}$$

$$= \frac{\text{Var}(X)}{\lambda E^2(X)} + \lambda$$

$$\geq \frac{\text{Var}(X)}{\lambda E^2(X)}$$

Put $\frac{1}{\lambda} = k$

$$\Rightarrow \frac{\text{Var}(S_N)}{E^2(S_N)} \geq k\frac{\text{Var}(X)}{E^2(X)}$$

Similarly, let $N \sim \text{Geo}(p)$

Then $G(s) = \frac{p}{1-qs}$

and $H(s) = \frac{p}{1-qP(s)}$

$$E(S_N) = \left.\frac{pqP'(s)}{(1-qP(s))^2}\right|_{s=1} = \frac{pqE(X)}{p^2} = \frac{q}{p}E(X)$$

$$\text{Var}(S_N) = \left.\frac{2pq^2(P'(s))^2}{(1-qP(s))^3}\right|_{s=1} + \left.\frac{pqP''(s)}{(1-qP(s))^2}\right|_{s=1} - \frac{q^2}{p^2}E^2(X) + \frac{q}{p}E^2(X)$$

$$= \frac{2pq^2E^2(X)}{p^3} + \frac{pqP''(1)}{p^2} - \frac{q^2}{p^2}E^2(X) + \frac{q}{p}E^2(X)$$

$$= \frac{q^2}{p^2}E^2(X) + \frac{q}{p}(P''(1) + P'(1) - (P'(1))^2) + \frac{q^2}{p^2}E^2(X) + \frac{q}{p}E^2(X)$$

$$= \frac{q}{p}\text{Var}(X) + \frac{q}{p^2}E^2(X)$$

$$\Rightarrow \frac{\text{Var}(S_N)}{E(S_N)} = \frac{\frac{q}{p}Var(X) + \frac{q}{p^2}E^2(X)}{\frac{q^2}{p^2}E^2(X)} = \frac{p}{q}\frac{\text{Var}(X)}{E(X)} + \frac{1}{q}$$

$$\geq \frac{p}{q}\frac{\text{Var}(X)}{E(X)}$$

$$= k\frac{\text{Var}(X)}{E(X)}; \quad k = \frac{p}{q}$$

1.10.1 Sum of a random number of continuous random variables

Laplace transforms:(LT): LT is a generalization of generating functions. Let $f(t)$ be a function of a positive real variable t. Then, LT of $f(t)$ is defined as

$$\overline{f(s)} = \int_0^\infty e^{-st} f(t)dt$$

for the range of values of s for which the integral exists.

We state and prove the following theorem.

Theorem 1.4: If X_i are i.i.d. random variables having LT's

$$F^*(s) = E(e^{-sX_i}) \quad \forall i$$

and if N is a random variable, independent of X_i's, having the p.g.f.

$$G(s) = \sum_n P(N=n)s^n = \sum_n g_n s^n$$

then the LT $F_{S_N}^*(s)$ of the sum $S_N = X_1 + X_2 + ... + X_N$ is given by

$$F_{S_N}^*(s) = G(F^*(s))$$

Proof: The p.d.f. $f_{S_N}(x)$ of S_N is given by

$$f_{S_N}(x) = P(x \le S_N \le x + dx)$$

$$= \sum_{n=0}^{\infty} P(N = n)P(x \le S_n \le x+dx)$$

$$= \sum_{n=0}^{\infty} g_n \; P(x \le S_n \le x + dx)$$

$$\Rightarrow \; F_{S_N}{}^{*}(x) = \int_0^{\infty} e^{-S_N t} f_{S_N}(t)dt$$

$$= \int_0^{\infty} e^{-\Sigma X_i t_i} \prod_i f_{X_i}(t_i)dt_i$$

$$= \sum_n g_n \left(\int_0^{\infty} e^{-X_i t_i} f_{X_i}(t_i)dt_i \right)^n$$

$$= \sum_n g_n (F^{*}(x))^n$$

$$= G(F^{*}(x))$$

Deduction: $$E(S_N) = \frac{d}{ds} F_{S_N}{}^{*}(s)\Big|_{s=1} = G'(F^{*}(s))F'^{*}(s)\,|_{s=1} = E(N)E(X)$$

Examples:

(24) **Random geometric sum of exponential variables:** Let X_i ~ exp(λ), i = 1,2,… and N ~ Geo(p). Further, let $S_N = X_1 + X_2 + ... + X_N$. Then

$$P(N = n) = pq^{n-1}$$

$$\Rightarrow \; G(s) = \frac{ps}{1-qs}$$

and

$$F_X{}^{*}(s) = \int_0^{\infty} \lambda e^{-st} e^{-\lambda t} dt$$

$$= \frac{\lambda}{\lambda+s} e^{-(s+\lambda)t}\Big|_0^{\infty}$$

$$= \frac{\lambda}{\lambda+s}$$

$$\Rightarrow \; H(s) = G(F_X{}^{*}(s)) = \frac{p\left(\frac{\lambda}{\lambda+s}\right)}{1-q\left(\frac{\lambda}{\lambda+s}\right)} = \frac{p\lambda}{s+p\lambda}$$

is the LT of exp(λp) distribution.

For an exponential variable, the characteristic function is given by

$$\phi_X(s) = \left(1 - \frac{\iota s}{\lambda}\right)^{-1}$$

(25) Let $X_i \sim \exp(a)$ and $N \sim \text{Pois}(\lambda)$. Then

$$F_{X_i}{}^*(s) = \frac{a}{a+s}$$

and $$G(s) = e^{\lambda(s-1)}$$

$$\Rightarrow H(s) = e^{\lambda\left(\frac{a}{a+s}-1\right)} = e^{\left(\frac{-\lambda s}{a+s}\right)}$$

Problems

1. Let $Y_t = \sum_{n=1}^{N} C_n e^{i\lambda_n t}$, $i = \sqrt{-1}$, where $\lambda_1, \lambda_2, ..., \lambda_N$ are real constants, and C_n's are uncorrelated random variables with zero mean and $\text{Var}(C_n) = \sigma_n{}^2$. Find if the process is stationary?

2. Let $\{X_t, t \geq 0\}$ be a continuous time stochastic process with independent increments. Further, let P ($X_0 = 0$) = 1. If $\varphi(\theta, t\text{-}u)$ is the characteristic function of a single increment, then show that the joint characteristic function of $X_{t_1}, X_{t_2}, ..., X_{t_n}$, $t_1 < t_2 < ... < t_n$ is

$$\varphi\left(\sum_{j=1}^{n} \theta_j, t_1\right)\varphi\left(\sum_{j=2}^{n} \theta_j, t_2 - t_1\right)...\varphi\left(\theta_n, t_n - t_{n-1}\right)$$

3. Consider a stochastic process $\{X_t, t \in T\}$, where

$$P(X = k) = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1,2... \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$$

Is the process a stationary process?

4. **Random telegraph signal process**: Let $\{X_t, t \geq 0\}$ be a stochastic process defined by $X(t) = Y(-1)^{N(t)}$, $t \geq 0$, where $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ and Y is a random variable independent of $N(t)$. Further, let $P(Y = \pm 1) = \frac{1}{2}$. Show that $\{X_t, t \geq 0\}$ is a covariance stationary process.

5. Let T_1, $T_2 \ldots T_N$ be the germination times of N seeds planted at time $t = 0$ and they are assumed to be i.i.d. with the density function $F(t)$ such that $F(0) = 0$ and $F(\infty) = 1-p$, p being the probability of a seed being fail to germinate. The average proportion of germination in an interval of infinitesimal length Δ is $\lambda > 0$. Find the probability distribution of X_t, the average number of seeds that will germinate in $(0,t)$.

6. Let $X(t) = \sin(wt)$, $w \sim U(0,2\pi)$. Then $\{X(t), t = 0,1,2\ldots\}$ is covariance stationary but not strictly stationary. What do you say about $\{X(t), t \geq 0\}$?

7. Let $X_n = \cos(nY)$, $n \geq 1$, and $Y \sim U(-\pi,\pi)$. Show that $\{X_n, n \geq 1\}$ is covariance stationary but not strictly stationary.

8. Let X be a random variable with generating function $P(S)$. Find the g.f.'s of $X+1$ and $2X$.

9. Let X be a random variable with generating function $P(S)$. Find the g.f.'s of

   (a) $P(X \leq n)$ (b) $P(X < n)$ (c) $P(X \geq n)$ (d) $P(X > n+1)$

   (e) $P(X = 2n)$

10. If a fair coin is tossed indefinitely, show that the event of two consecutive heads is certain to occur. Find the expected number of trials for the event to occur. What can you say about event of three, four or n consecutive heads?

11. An event, which is certain to occur, is called a persistent event. Are the events defined in earlier problem persistent?

12. In a sequence of Bernoulli trials, let u_n be the probability that the combination *SFS* occurs for the first time at the nth trial, i.e., a success at n-2th trial, followed by a failure at n-1th trial and a success at n^{th} trial. Find the generating function, the mean and the variance of $\{u_n\}$.

13. Let ω_n be the number of ways in which the score n can be obtained by throwing a die any number of times. Find the generating function of $\{\omega_n\}$.

14. **Converse of ageing property of geometric distribution:** If T is a non-negative discrete random variable such that

$$P(T > x + y \mid T > x) = P(T > y)$$

   for all non-negative integers x and y, Then , show that T has a geometric distribution.

15. Let X have a zero-truncated Poisson distribution with zero class missing, i.e.,

$$P(X = k) = (e^a - 1)^{-1} \frac{a^k}{k!}, \quad k = 1, 2, 3, \ldots$$

Find the p.g.f., the mean and the variance of X.

16. This is the famous **animal-trapping problem**. If g_n = P (a species is of size n}, and all the animals have the same probability p of being caught, then the number of trapped animals of a species in the sample is a variable S_N with gf $g(q + ps)$.

17. In a sequence of N Bernoulli trials, where N ~ Pois (λ), show that the number of successes and failures are stochastically independent.

18. Let $S_N = X_1 + X_2 + ... + X_N$, $N \sim Pois(\lambda)$. If X_i ~ Geo (p), find the generating function, the mean and the variance of S_N.

19. Let $X \sim Pois(\lambda)$ where λ(>0) is a gamma variable. Then show that

$$P(X = k) = \frac{(k+n)!}{n!(k+1)!}\left(\frac{1}{2}\right)^{k+n}; k = 0,1,2..., \quad n = 0,1,2...$$

20. Consider a sequence of independent random variables X_1, X_2… distributed identically. Further let $S_n = X_1 + X_2 + ... + X_n$, where n is a positive integer. Find
    (i) the joint characteristic function of $X_1, X_2 \dots X_n$;
    (ii) the joint characteristic function of $S_1, S_2 \dots S_n$;
    (iii) the joint characteristic function of $Y_1, Y_2 \dots Y_n$; where, $Y_k = X_{k+1} - X_k$, if

    (a) $X \sim N(0, \sigma^2)$

    (b) $X \sim Pois(\lambda)$

    (c) $X \sim \exp(-\lambda)$