Chapter 8

Replacement Theory

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Key words: Replacement theory, time value of money, sudden failure, group replacement, mortality and preventive replacement.

Suggested readings:

1. Gupta P.K. and Mohan M. (1987), **Operations Research and Statistical Analysis**, Sultan Chand and Sons, Delhi.
2. Johnson R.D. and Bernard R.S. (1977), **Quantitative Techniques for Business Decisions**, Prentice hall of India Private Limited
3. Swarup K., Gupta P.K. and Mohan M. (2001), **Operations Research**, Sultan Chand and Sons, Delhi.

8.1 Introduction

The study of replacement is concerned with the situations that arise when some items such as equipment need replacement due to changes in their performance. This change may either be gradual or all of a sudden.

Broadly speaking, the requirement of a replacement may be in any of the following situations:

(i) An item fails and does not work at all or the item is expected to fail shortly.

(ii) An item deteriorates and need expensive maintenance.

(iii) A better design of the equipment is available.

(iv) It is economical to replace equipment in anticipation of costly failure.

In this chapter, we are interested in the first two situations. Third situation has been dealt when we studied the pay-off criteria.

When studying the problem of replacement, we may or may not consider the time value of money.

8.2 Replacement of equipment that deteriorates gradually

Generally, the cost of maintenance and repairing of certain equipments increases with time and ultimately the cost may become so high that it is more economical to replace theses equipments with new ones. If the productivity of equipment decreases with time, this may also be considered as a failure. At this point a replacement is justified.

The costs associated with aging increase at an increasing rate whereas the resale value of the equipment decreases at increasing rate. The decreasing resale value results in increasing depreciation, which is the difference between the purchase price and the resale value. The depreciation of the item increases at a decreasing rate.

The optimal replacement policy for such items is to replace the equipment at a point where the total cost curve intersects the total depreciation curve

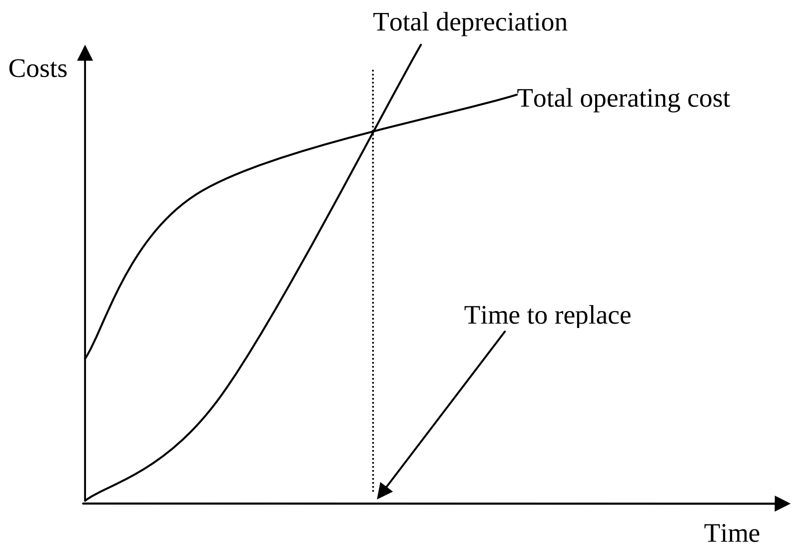

Fig. 8.1(i)

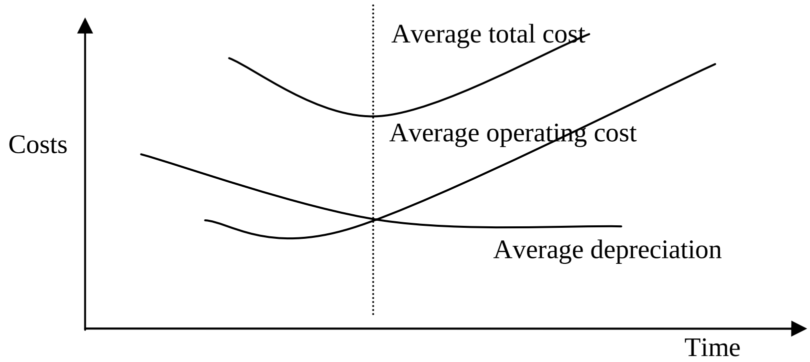

Fig. 8.1(ii)

(a) Time value of money does not change

If the value of money does not change with time, then the user of the equipment does not need to pay interest on his investments. We wish to determine the optimal time to replace the equipment.

We make use of the following notations:

C: Capital cost of the equipment

S: Scrap value of the equipment

n: Number of years that the equipment would be in use.

$f(t)$: Maintenance cost function.

$A(n)$: Average total annual cost.

Two possibilities are there

(i) Time *t* is a continuous random variable

In this case the deterioration of the equipment is being monitored continuously. The total cost of the equipment during n years of use is given by

$$TC = \text{Capital cost - Scrap value + Maintenance cost}$$

$$= C - S + \int_0^n f(t)dt$$

$$\Rightarrow A(n) = \frac{1}{n}TC = \frac{C - S}{n} + \frac{1}{n}\int_0^n f(t)dt$$

For minimum cost, $\frac{d}{dn}A(n) = 0$

$$\Rightarrow -\frac{C - S}{n^2} - \frac{1}{n^2}\int_0^n f(t)dt + \frac{1}{n}f(n) = 0$$

$$\Rightarrow f(n) = \frac{C - S}{n} + \frac{1}{n}\int_0^n f(t)dt = A(n)$$

and $\frac{d^2A(n)}{dn^2} \geq 0$ at $f(n) = A(n)$

i.e., when the maintenance cost becomes equal to the average annual cost, the decision should be to replace the equipment.

(ii) Time *t* is a discrete random variable

In this case

$$A(n) = \frac{1}{n}TC = \frac{C - S}{n} + \frac{1}{n}\sum_0^n f(t)$$

A (n) is minimum when

$$A(n+1) \geq A(n) \text{ and } A(n-1) \geq A(n)$$

$$\text{or,} \quad A(n+1) - A(n) \geq 0 \text{ and } A(n) - A(n-1) \leq 0$$

$$A(n+1) - A(n) = \frac{1}{n+1}\left(C - S + \sum_{t=1}^{n} f(t)\right) + \frac{1}{n+1} f(n+1) - A(n)$$

$$= \frac{n}{n+1} A(n) + \frac{1}{n+1} f(n+1) - A(n) \geq 0$$

$$\Rightarrow \quad f(n+1) \geq A(n)$$

Similarly

$$A(n) - A(n-1) \leq 0$$

$$\Rightarrow \quad f(n) \leq A(n-1)$$

Thus the optimal policy is

Replace the equipment at the end of n years if the maintenance cost in the (n+1) th year is more than the average total cost in the nth year and the nth year's maintenance cost is less than previous year's average total cost.

Example 1: A firm is considering replacement of a machine, whose cost price is Rs, 1,20,000, and the scrap value is Rs. 20,000. The running (maintenance and operating) costs of the machine are as follows:

Table 8.1

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	2,000	5,000	8,000	12,000	18,000	25,000	32,000	40,000

When should the machine be replaced?

Sol:

$$C = \text{Rs. } 1,20,000$$

$$S = \text{Rs. } 20,000$$

$$f(n) = \text{Running cost in the } n^{th} \text{ year.}$$

$$TC = C - S + \sum_{0}^{n} f(t)$$

$$= 1{,}00{,}000 + \sum_{0}^{n} f(t)$$

We prepare the following table

Table 8.2

Year of service (n)	Running cost (Rs.) $f(n)$	Cumulative running cost (Rs.) $\sum_{n} f(n)$	$TC = C - S + \sum_{0}^{n} f(t)$ (Rs.)	$A(n) = \frac{1}{n} TC$ (Rs.)
1	2,000	2,000	1,02,000	1,02,000
2	5,000	7,000	1,07,000	53,500
3	8,000	15,000	1,15,000	38,334
4	12,000	27,000	1,27,000	31,750
5	18,000	45,000	1,45,000	29,000
6	25,000	70,000	1,70,000	**28,334**
7	32,000	1,02,000	2,02,000	28,857
8	40,000	1,42,000	2,42,000	30,250

Thus *A* (6) = Rs. 28,334 is minimum. Hence replacement should be done after every sixth year.

Example 2: A printing machine costs Rs. 40,000 when new. The following table gives the expected operating costs, expected production (per 100 pages) and the resale value of the machine

Table 8.3

Age	1	2	3	4	5	6
Operating costs (Rs.)	5,000	7,000	8,000	10,000	12,500	15,000
Production	20,000	20,000	19,000	18,000	15,000	13,500
Resale value (Rs.)	30,000	25,000	19,000	15,000	12,000	10,000

The production cost per unit of production is Rs. 15. Find the optimal replacement policy.

Sol:

Table 8.4 (Rs.)

Age	Total O.C.	Total production cost	Depreciation	Total cost	Average cost
1	5,000	0	10,000	15,000	15,000
2	7,000	0	15,000	22,000	**11,000**
3	8,000	$1000 \times 15 = 15{,}000$	21,000	44,000	14667
4	10,000	30,000	22,000	62,000	15500
5	12,500	75000	25,000	1,12,500	22500
6	15,000	97500	26,500	1,39,000	23167

The machine should be replaced after every 2 years.

Example 3: A factory owner finds from his past records that the costs per year of running a machine whose purchase price is Rs. 60,000 are as follows

Table 8.5

Year	1	2	3	4	5	6	7
Operating costs (Rs.)	10,000	12,000	14,000	18,000	23,000	28,000	34,000
Resale value (Rs.)	30,000	15,000	7,500	3,750	2,000	2,000	2,000

The owner has three machines, two of which are two-years old and the third is one year old. He is now considering a new machine with 50% more capacity than one of the old ones. The price of this machine is Rs. 8, 000. The cost estimates for the new machine are as follows

Table 8.6

Year	1	2	3	4	5	6	7
Operating costs (Rs.)	12,000	15,000	18,000	24,000	31,000	40,000	50,000
Resale value (Rs.)	40,000	20,000	10,000	5,000	3,000	3,000	3,000

Assume that the loss of flexibility due to fewer machines is insignificant and he has sufficient work in any of the decisions, what should be the optimal policy?

Sol: For the first type of trucks, the average annual costs can be computed as follows

Table 8.7

Age	Total O.C. (Rs)	Depreciation (Rs.)	Total cost (Rs)	Average cost (Rs)
1	10,000	30,000	40,000	40,000
2	22,000	45,000	67,000	33,500
3	36,000	52,500	88,500	29,500
4	54,000	56,250	1,10,250	27562.5
5	77,000	58,000	1,35,000	**27,000**
6	1,05,000	58,000	1,63,000	27167
7	1,39,000	58,000	1,97,000	28143

For the new machine, the average annual cost has been calculated in the following schedule

Table 8.8

Age	Total O.C. (Rs)	Total capital cost (Rs) (Total depreciation)	Total cost (Rs)	Average cost (Rs)
1	12,000	40,000	52,000	52,000
2	27,000	60,000	87,000	43,500
3	45,000	70,000	1,15,000	38,333
4	69,000	75,000	1,44,000	36,000
5	1,00,000	77,000	1,77,000	**35,400**
6	1,40,000	77,000	2,17,000	31,167
7	1,90,000	77,000	2,67,000	38,143

(i) If the first type of machines is opted, then replacement should be after every five weeks.

(ii) If the second type of machines is opted, then replacement should be after every five weeks.

(iii) Average annual cost of old machine = Rs. 27,000

Average annual cost of new machine = Rs. 35,400

Equivalent cost of old machine = $35,400 \times \frac{2}{3}$ = Rs. 23,600

Thus old machines should be replaced by new ones.

(iv) Three old machines are to be replaced by two new machines when the joint running cost of the old machines is more then the average yearly cost of two new machines (i.e., $35,400 \times 2 =$ Rs. 70,800).

The total cost of old machines is calculated below

Table 8.9

Year	Total annual cost of one old machine (Rs.)	Total annual cost of three given old machine (Rs.)
1	40,000	
2	67,000-40,000 = 27,000	
3	21,500	$2 \times 21,500 + 27,000 = 70,000$
4	21,750	**65,000**
5	24,750	71,250

The old machines must be replaced by new machines two years from now.

(b) Time value of money changes

In this case, the investor is paying interest on the money that has been invested. We assume that

(i) The maintenance costs are incurred at the beginning of time periods; and

(ii) The maintenance costs increase with time.

Let the money carry an interest rate of *r*% per year, i.e., one rupee in *n* years time is equivalent to Rs. $(1+r)^{-n}$ today.

$(1+r)^{-n}$ is called the **present worth factor of rupee one spent *n* years after now**.

$(1+r)^{n}$ is the **payment compound amount factor of rupee one** spent in *n* years time.

Let

C: Initial cost of the equipment

R_n: Running cost of the equipment in year *n.*

ν: Rate of interest = $(1+r)^{-1}$

V_n: Future discounted costs associated with the policy of replacing the equipment at the end of each n years.

Then the present value of V_n is

$$\begin{aligned} V_n &= \left\{(C+R_0)+\nu R_1+\nu^2 R_2+\ldots+\nu^{n-1}R_{n-1}\right\} \\ &\quad + \left\{(C+R_n)\nu^n+\nu^{n+1}R_{n+1}+\nu^{n+2}R_{n+2}+\ldots+\nu^{2n-1}R_{2n-1}\right\} \\ &\quad + \ldots \\ &= \left\{C+\sum_{k=0}^{n-1}\nu^k R_k\right\}\sum_{k=0}^{\infty}\left(\nu^n\right)^k \\ &= \frac{\left\{C+\sum_{k=0}^{n-1}\nu^k R_k\right\}}{1-\nu^n} \end{aligned}$$

Now, V_n is minimum when

$$V_{n+1} \geq V_n \text{ and } V_{n-1} \geq V_n$$

Now,

$$\begin{aligned} V_{n+1} - V_n &= \frac{\left\{C+\sum_{k=0}^{n-1}\nu^k R_k\right\}}{1-\nu^{n+1}} - V_n \\ &= \frac{\nu^n\left(R_n-(1-\nu)V_n\right)}{1-\nu^{n+1}} \end{aligned}$$

Similarly,

$$V_n - V_{n-1} = \frac{\nu^{n-1}\left(R_{n-1}-(1-\nu)V_n\right)}{1-\nu^{n-1}}$$

Now, $\nu < 1$, ν being depreciation factor,

$$\Rightarrow 1-\nu > 0$$

$$\Rightarrow \frac{\nu^n}{1-\nu^{n+1}} > 0$$

$$\Rightarrow V_{n+1} - V_n > 0 \Rightarrow R_n > (1-\nu)V_n$$

$$\text{and } V_{n-1} - V_n > 0 \Rightarrow R_{n-1} < (1-\nu)V_n$$

$$\Rightarrow R_{n-1} < (1-\nu)V_n < R_n$$

$$\Rightarrow R_{n-1} < W_n = \frac{(C+R_0)+\nu R_1 + \nu^2 R_2 + \ldots + \nu^{n-1}R_{n-1}}{1+\nu+\nu^2+\ldots+\nu^{n-1}} < R_n$$

The expression, which lies between R_n and R_{n-1} is called the **weighted average cost** of all the previous n years with weights being $1, \nu, \nu^2, \ldots$ and ν^{n-1} respectively.

Thus the optimal replacement policy is:

(a) ***Do not replace the equipment if the next period's operating cost is less than the weighted average of previous costs.***

(b) ***Replace the equipment if the next period's operating cost is greater than the weighted average of previous costs.***

$$For\ \nu = 1, \quad R_{n-1} < V_n < R_n$$

Example 4: A manufacturer is to make a choice between two machines, say, *A* and *B*, which are priced at Rs. 50,000 and Rs. 25,000 respectively. The annual running costs for machine *A* are Rs. 8,000 for first five years after which the costs increase per year by Rs. 2,000. Machine *B*, which has the same capacity as the machine *A*, will have a running cost of Rs. 12,000 for first six years, and after that would increase by Rs. 2,000 per year.

If the money is worth 10% per year, which machine should be purchased? Assume that the scrap value of the two machines is nil.

Sol:

$$r = .10$$

$$\Rightarrow v = \frac{1}{1+.10} = 0.9091$$

We prepare the following tables:

Table 8.11: For machine A

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1}R_{n-1}$	$C+\sum v^{k-1}R_{k-1}$	$\sum v^{k-1}$	W_n
1	8000	1.0000	8000	58000	1.0000	58000
2	8000	0.9091	7272.80	57272.80	1.9091	34190.35
3	8000	0.8264	6611.20	56611.20	2.7355	26278.19
4	8000	0.7513	6010.40	56010.40	3.4868	22339.8
5	8000	0.6830	5464	55464	4.1698	19990.98
6	10000	0.6209	6209	56209	4.7907	18696.1
7	12000	0.5645	6774	56774	5.3552	17990.25
8	14000	0.5132	7184.80	57184.80	5.8684	17641.3
9	16000	0.4665	7464	57464	6.3349	**17520.43**
10	18000	0.4241	8482	58482	6.7590	17525.53

Table 8.12: For machine B

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1}R_{n-1}$	$C+\sum v^{k-1}R_{k-1}$	$\sum v^{k-1}$	W_n
1	12000	1.0000	12000	37000	1.0000	37000
2	12000	0.9091	10909.20	47909.20	1.9091	25095.18
3	12000	0.8264	9916.80	57826	2.7355	21139.10
4	12000	0.7513	9015.60	66841.60	3.4868	19169.90
5	12000	0.6830	8196	75037.60	4.1698	17995.49
6	14000	0.6209	8692.60	83730.20	4.7907	17477.65
7	12000	0.5645	6774	90504.20	5.3552	16900.25
8	16000	0.5132	8211.20	98715.40	5.8684	**16821.52**

9	18000	0.4665	8397	107112.40	6.3349	16908.30
10	20000	0.4241	8482	115594.40	6.7590	17102.29

For machine *A*, the running cost in the ninth year of operations is the least so it should be replaced after every ten years.

For machine *B*, the running cost in the eighth year of operations is the least so it should be replaced after every eight years.

Further the weighted average cost in ten years of machine *A* is Rs. 17505.53, whereas the weighted average cost in eight years of machine *B* is Rs. 16821.52. So machine *B* should be purchased.

8.3 Replacement of equipments that fail suddenly

Since the exact failure time is difficult to predict for those equipments, which fail suddenly, in such cases we try to obtain the probability distribution of failures. It is assumed that the failures occur at the end of the period, say, *t*. Then the objective is to determine the value of *t* that minimizes the total cost involved in the replacement.

In this case, two types of replacement are involved. In first replacement, equipment is replaced as and when it fails.

In second type of replacement, equipment may be replaced even before it fails. This kind of replacement is undertaken in those situations when failure of equipment results in huge monetary losses (e.g. electricity transformers) or is fatal in nature (e.g. a pace maker).

We define the following replacement policies

Individual replacement policy: An item is replaced immediately after its failure. This policy is called as the individual replacement policy

Group replacement policy: Under this policy, decisions are taken as to when the items must be replaced irrespective of the fact that the items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced. For example, electricity bulbs on a road are replaced after some time even if they have not failed. But an individual failed bulb is replaced during the interval between two successive group replacements also.

We define the following notations:

$M(t)$ = the number of survivors at any time t.

N = initial number of items.

$p(t)$ = P(of failure during the time-period t)

$$= \frac{M(t-1)-M(t)}{M(t-1)}$$

$p_s(t)$ = P(of survival till time-point t)

$$= \frac{M(t)}{N}$$

We have the following results:

Theorem 8.1: (Mortality) A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no immigrations or emigrations. Then the age distribution ultimately becomes stable and the number of deaths per unit time becomes constant, which is given by the size of the total population divided by the mean age of deaths.

Proof: Let the maximum period of survival is k

Define

$f(t)$ = number of births at time t;

$p(x)$ = P(of daeth at time x).

Then $$\sum_{x=0}^{k} p(x) = 1$$

And $$E(\text{deaths till time } t) = \sum_{x=0}^{k} f(t-x)\,p(x) = f(t+1)$$

Now, $$f(t+1) = \sum_{x=0}^{k} f(t-x)\,p(x) \tag{8.1}$$

This is a difference equation in t. To find a solution to this equation, put $f(t) = A\alpha^t$, where A is an arbitrary constant and α is a number between 0 and 1.

$$\therefore A\alpha^{t+1} = \sum_{x=0}^{k} A\alpha^{t-x} p(x)$$

$$= A\left(\alpha^t p(0) + \alpha^{t-1} p(1) + ... + \alpha^{t-k} p(k)\right)$$

$$\Rightarrow \quad \alpha^{t+1} = \alpha^t p(0) + \alpha^{t-1} p(1) + ... + \alpha^{t-k} p(k)$$

Let $t = k$. Then $\alpha^{k+1} - \alpha^k p(0) - \alpha^{k-1} p(1) - ... - p(k) = 0$ (8.2)

This is a linear homogenous equation of degree k+1 so it must have k+1 roots.

For $\alpha = 1$, L.H.S. of (8.2) = R.H.S. so 1 is one of the roots of the equation.

Let the other k roots be $\alpha_1, \alpha_2, ..., \alpha_k$. Then a general solution to (8.1) is given by

$$f(t) = A_0 1^t + A_1\alpha_1^t + ... + A_k\alpha_k^t$$

where A_i's are arbitrary constants to be determined.

Also $|\alpha| < 1 \Rightarrow |\alpha_i| < 1 \; \forall \; i$. Letting $t \to \infty$, we have

$$\lim_{t\to\infty} f(t) = A_0$$

Now, our job is to determine A_0. For this, define

$$q(x) = P(\text{survival upto } x \text{ or more years of age})$$

$$= 1 - P(\text{death before age } x)$$

$$= 1 - \sum_{t=0}^{x-1} p(t).$$

Obviously, $q(0) = 1$

Since births are immediately replaced by deaths, i.e., A_0 is the long run number of births as well as the number of deaths, so we have

$$E(\text{survivals upto age } x) \;=\; A_0 q(x)$$

$$\Rightarrow \;\; N = A_0 \sum_{x=0}^{k} q(x)$$

$$\Rightarrow \;\; A_0 = \frac{N}{\sum_{x=0}^{k} q(x)}$$

Finally, we have to determine the denominator of this expression. For this, consider

$$\Delta x \;=\; (x+1) - x; \text{ and}$$

$$\sum_{x=a}^{b} f(x)\Delta h(x) \;=\; f(b+1)h(b+1) - f(a)h(a) - \sum_{x=a}^{b} h(x+1)\Delta f(x)$$

Then we have

$$\sum_{x=0}^{k} q(x) \;=\; \sum_{x=0}^{k} q(x)\Delta x$$

$$= \;(k+1)q(k+1) - 0 \times q(0) - \sum_{x=0}^{k} (x+1)\Delta q(x)$$

$$= \;(k+1)q(k+1) - \sum_{x=0}^{k} (x+1)\Delta q(x)$$

But, $$q(k+1) \;=\; 1 - \sum_{i=0}^{k} p(i) \;=\; 1-1 \;=\; 0$$

And $$\Delta q(x) \;=\; q(x+1) - q(x) \;=\; -p(x)$$

$$\therefore \sum_{x=0}^{k} q(x) \;=\; (k+1)q(k+1) - \sum_{x=0}^{k} (x+1)(-p(x))$$

$$= \;\sum_{x=0}^{k} (x+1)p(x)$$

$$= \;\text{Mean age at death}$$

Hence, we have $$A_0 \;=\; \frac{N}{\text{Mean age at death}}$$

Hence the result.

Example 5: A company manufactures automobile batteries at a cost of Rs. 1000 each. Battery life is subject to the following mortality schedule

Table 8.13

Age (months) (i)	Probability of failure in next month (p_i)	Age (months) (i)	Probability of failure in next month (p_i)
0	0.050	16	0.000
1	0.000	17	0.100
2	0.000	18	0.100
3	0.000	19	0.100
4	0.000	20	0.100
5	0.000	21	0.010
6	0.000	22	0.010
7	0.000	23	0.015
8	0.000	24	0.015
9	0.000	25	0.020
10	0.000	26	0.025
11	0.000	27	0.030
12	0.000	28	0.035
13	0.000	29	0.040
14	0.000	30	0.710
15	0.000		

The company has a guarantee policy under which if a battery fails during the first month after purchase, either a refund of the full price or a new battery is made; if it fails during the second month, a refund of 29/30 of the full price is made; during third month, this refund is 28/30 of the full price and so on until 30th month when the refund is 1/30 of the full price. At what unit price should the batteries be sold so that on average will the company break-even?

Sol: Let the break-even price is Rs. P per battery and

$$p_i = P(\text{a new battery fails during the } i+1^{st} \text{ month})$$

The average refund on the failed batteries is given by

$$\left(1\times p_1 + \frac{29}{30}\times p_2 + \frac{28}{30}\times p_3 + \ldots + \frac{1}{30}\times p_{30}\right)P = 0.0908P$$

The break-even price P is the sum of the expected refund and the factory cost of the battery. Therefore

$$P = 1000 + 0.0908P$$

$$\Rightarrow 0.9092P = 1000$$

$$\Rightarrow P = \frac{1000}{0.9092} = Rs.1100$$

Theorem 8.2: (Group replacement) Let all the items of a system be replaced after a time interval‘t’ with the provision that an individual replacements can be made if and when any item fails during this time period. Then the optimal policy of replacement is

(i) Group replacement must be made at the end of the t^{th} time period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of *t* periods.

(ii) Group replacement must not be made at the end of period *t* if the cost of individual replacement at the end of the period *t* -1 is less than the average cost per unit time period through the end of *t* periods.

Proof: Let

N = Total number of items in the system.

C_1 = the cost of replacing an item in group.

C_2 = the cost of replacing an item individually.

$C(t)$ = Total cost function of group replacement after time period *t*.

$f(t)$ = Total number of failuers during time period *t*.

Then, $$C(t) = NC_1 + C_2 \sum_{x=0}^{t-1} f(x)$$

$\Rightarrow$ Average cost of group replacement per unit time during an interval of *t* units is given by

$$A(t) = \frac{C(t)}{t} = \frac{NC_1 + C_2 \sum_{t=0}^{i-1} f(x)}{t}$$

We want to minimize $\frac{C(t)}{t}$. In that case whenever

$$\frac{C(t-1)}{t-1} > \frac{C(t)}{t} \text{ and } \frac{C(t+1)}{t+1} > \frac{C(t)}{t}$$

We replace all items after time *t*.

Now,

$$\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$$

$$\Rightarrow \quad C_2 f(t) > \frac{C(t)}{t}$$

$$\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$$

$$\Rightarrow \quad C_2 f(t-1) < \frac{C(t)}{t}$$

$$\therefore \quad tC_2 f(t-1) < C(t) < tC_2 f(t)$$

$$\text{or,} \quad tf(t-1) = \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < tf(t) - \sum_{x=0}^{t-1} f(x)$$

Hence the optimal replacement policy is

(i) Group replacement must be made at the end of the t^{th} time period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of *t* periods.

(ii) Group replacement must not be made at the end of period *t* if the cost of individual replacement at the end of the period *t* -1 is less than the average cost per unit time period through the end of *t* periods.

Example 6: The following failure rates have been observed for a certain type of electronic equipments in a digital system

Table 8.14

End of the week	1	2	3	4	5	6	7	8
Probability of failure till date	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The cost of replacement of an individual failed component is Rs. 125. The decision is made to replace all these components simultaneously at fixed intervals, and to replace the individual components as they fail in service. If the cost of group replacement is Rs. 30 per unit, what is the best interval between group replacements?

Sol: Let there be 1,000 components in use, i.e. N = 1,000.

$$p_i = P(\text{ a components fails during } i^{th} \text{ week})$$

and $$N_i = \text{number of replacements at the end of the week, } i = 1, 2, \ldots 8.$$

We calculate N_i as follows:

Table 8.15

t	Probability of failure till time t	$p_i = C_i - C_{i-1}$	$N_i = \sum_{t=0}^{i-1} N_t p_{i-t}$ $N_0 = 1000$
1	0.05	0.05	50
2	0.13	0.08	82
3	0.25	0,12	128
4	0.43	0.18	199
5	0.68	0.25	289
6	0.88	0.20	272
7	0.96	0.08	194
8	1.00	0.04	195

Thus N_i oscillates till the system acquires a steady state. The expected life of the system is given by

$$\text{Expected life } = \sum_{i=1}^{8} ip_i = 4.62 \text{ weeks.}$$

$$\text{Expected number of failures per week } = \frac{1000}{4.62} \approx 216$$

$$\Rightarrow \text{ cost of individual replacement} = 216 \times 125 = \text{Rs. } 27{,}000$$

For group replacement

Table 8.16

t	Individual replacement *N* (*t*)	Total cost of replacement (individual + group) (Rs.) $N\ (t) \times 125 + 30 \times 1000$	Average cost (Rs.)
1	50	36250	36250
2	132	46500	23250
3	268	63500	**21167**
4	459	87375	21843.75

The optimum interval for group replacement is 3 weeks. The group replacement cost for this interval is less than the individual replacement cost, so it is better to adopt group replacement.

Example 7: At $t = 0$, all the items in a system are new, each of which has a constant probability p of failure before the end of the first month of life and a probability q (= 1-p) of failure before the end of the second month. If all the items are replaced as they fail, show that the expected number of failures f (x), at the end of the month x is given by

$$f(x) = \frac{N}{1+q}\left(1-(-q)^{x+1}\right); x = 1, 2 \text{ ,}$$

where N is the number of items in the system.

If the cost per unit of the individual replacement is C_1, and the cost of group replacement per item is C_2, find the conditions under which

(a) A group replacement policy at the end of each month is the most profitable;

(b) No group replacement policy is better than the individual replacement policy.

Sol: Let

$$N_1 = \text{the number of items expected to fail at the end of the first month}$$

$$= N_0 p = N(1-q)$$

$$N_2 = \text{the number of items expected to fail at the end of the second month}$$

$$= N_0 q + N_1 p$$

$$= Nq + N(1-q)^2$$

$$= N(1-q+q^2)$$

In general,

$$N_k = N(1-q+q^2-q^3+...+(-q)^k)$$

$$\Rightarrow \quad N_{k+1} = N_{k-1}q + N_k p$$

$$= N(1-q+q^2-q^3+...+(-q)^{k-1})q + N(1-q+q^2-q^3+...+(-q)^k)(1-q)$$

$$= N(1-q+q^2-q^3+...+(-q)^{k+1})$$

$$\Rightarrow \quad f(x) = N(1-q+q^2-q^3+...+(-q)^x)$$

$$= N\left(\frac{1-(-q)^{x+1}}{1+q}\right)$$

$$\lim_{x\to\infty} f(x) = \frac{N}{1+q}$$

$$= \frac{\text{Total number of items in the system}}{\text{mean age}}$$

is the steady-state expectation.

(1) For group replacement policy at the end of each month, cost of replacement is NC_2.

(2) For group replacement policy at the end of every second month, cost of replacement is NC_2 + NpC_1

$$\Rightarrow \quad \text{average cost per month} = \frac{NC_2 + NpC_1}{2}$$

(3) Average life of an item = $1\times p + 2\times q \;=\; 1+q$

$$\Rightarrow \quad \text{cost of individual replacement} = \frac{NC_1}{1+q}$$

(a) A group replacement at the end of the first month will be better than the individual replacement, if the total cost of the group replacement is less than the average monthly cost of the individual replacement, i.e.

$$N(1-q)+NC_2 \;<\; \frac{NC_1}{1+q}$$

$$\Rightarrow \quad C_2 \;<\; \frac{C_1 q^2}{1+q}$$

For a group replacement at the end of the every second month, the total cost of total replacement will be

$$(N_1+N_2)C_1+NC_2 \;=\; N(2-2q+q^2)\;C_1+\;NC_2$$

$$\Rightarrow \quad \text{Average monthly cost of group replacement} \;=\; \frac{N(2-2q+q^2)\;C_1+\;NC_2}{2}$$

i.e., a group replacement at the end of the every second month will be better than the individual replacement, if the total cost of the group replacement is less than the average monthly cost of the individual replacement, i.e.

$$\frac{N(2-2q+q^2)\;C_1+\;NC_2}{2} \;<\; NC_1(1+q)$$

$$\text{or,} \quad C_2 \;<\; \frac{q^2(1-q)C_1}{(1+q)}$$

(b) For individual replacement policy to be better than any of the group replacement policies, we must have

$$C_2 \;>\; \frac{q^2C_1}{(1+q)} \quad \text{and} \quad C_2 \;>\; \frac{q^2C_1(1-q)}{(1+q)}$$

$$\text{or,} \quad C_1 \;<\; \frac{C_2(1+q)}{q^2} \quad \text{and} \quad C_1 \;<\; \frac{1+q}{q^2(1-q)}$$

$$\text{But} \quad q < 1 \quad \Rightarrow \quad \frac{1+q}{q^2} < \frac{1+q}{q^2(1-q)}$$

$$\Rightarrow \quad C_1 < \frac{1+q}{q^2} C_2$$

8.4 Preventive replacement

If the cost of the failure of the equipment is much more than the cost of its replacement, e.g. a pacemaker or an electronic chip in an aviation system, then it is advisable to replace that equipment before it fails. We now derive the optimal replacement policy for preventive replacement.

C_R = Cost of replacement.

C_F = Cost of failure (including replacement).

T = the random variable denoting the life of the equipment.

$F_T(t) = P(T \geq t) = P$(the equipment will not fail till time point t).

$f(t) = P$(of failure in time period $t \mid T \geq t$).

Then $F_T(t) = F_T(t-1)\,\left(1\text{-}f(t)\right)$

Suppose that the replacement is made at the beginning of an interval.

Let the replacement policy be to replace the equipment after every T_r period if the item has not failed earlier, i.e.,

$$T_r = \max t$$

Then the expected life of the equipment is

$$E(T) = \sum_{t=1}^{T-1} tF(t-1)\,f(t) + T\left(1 - F(t-1)\right)$$

In an interval of length, say, ω, the number of equipment required is $\dfrac{\omega}{E(T)}$.

Thus the expected cost of replacement is

$$= \frac{\omega}{E(T)}\left(1 - F_T(t-1)\right)C_F + \frac{\omega}{E(T)} F_T(t-1)C_R$$

$$\Rightarrow \; AC(T) = \frac{\left(1 - F_T(t-1)\right)C_F + F_T(t-1)C_R}{E(T)}$$

Where *AC (T)* is the average cost curve. Then the *T*, which minimizes *AC (T)*, is the optimal time to replace the equipment.

Example 8: A small component in a machine costs Rs. 50 and it takes, on average, 10 minutes to replace it. However, in 10 minutes, the machine can produce goods of worth Rs. 300. the probability of the failure of the component increases with the usage so that after some time of usage, it is advisable to replace the component. The cost of the replacement of the component is Rs. 25. The probability distribution of the failures is given in the following table:

Table 8.17

Weeks	1	2	3	4	5
f(t)	0.01	0.05	0.10	0.15	0.25

Find the optimal time to replace the component.

Sol:

C_R = Cost of replacement = Rs. 25.

C_F = Cost of failure = Rs.300 + Rs.25 = Rs. 325.

Table 8.18

Weeks	1	2	3	4	5
f(t)	0.01	0.05	0.10	0.15	0.25
$F_T(t) = F_T(t\text{-}1)(1\text{-}f(t))$	0.99	0.941	0.847	0.720	0.540
E (T)	1	1.990	2.931	3.778	4.498
$\left(1-F_T(t-1)\right)C_F + F_T(t-1)C_R$	25	28	42.7	70.9	109
$AC(T)$	25	**14**	14.23	17.73	21.8

The optimal time to replace the chip is at the beginning of the second week.

Problems

1. Following data has been collected from the records regarding the running cost of a machine priced at Rs. 6,00,000.

Table 8.19

Year	1	2	3	4	5	6	7
Operating costs (Rs.)	10,000	12,000	14,000	25,000	35,000	50,000	80,000
Resale value (Rs.)	4,00,000	2,65,000	1,75,000	1,25,000	90,000	60,000	45,000

Determine the optimal time of replacement.

2. The cost of a new machine is Rs. 1,00,000. Compute the optimal time to replace when the costs associated with the machine are given below:

Table 8.20

Age (Years)	1	2	3
Operating costs (Rs.)	50,000	70,000	90,000
Resale value (Rs.)	80,000	65,000	50,000

Assume that the repairs are made at the end of a period only if the machine is to be retained and not necessary if the machine is to be sold. Assume that the cost of capital is 10%.

3. A population of N individuals is subject to a given mortality law per unit of time. The deaths are immediately replaced by births at the end of the interval. No individual can survive more than r periods. Show that the distribution of the number of deaths ultimately stabilizes to

$$\frac{N}{Average\ life}$$

4. Consider the following replacement schedule of a component in an electronic gadget

Table 8.21

Hours in use	300	600	900	1200
Probability of failure	0.05	0.30	0.75	1.00

The cost of the replacement of the part is Rs. 500 whereas the failure would cost Rs. 3,000. What should be the optimal replacement policy?

5. A truck fleet owner owns 50 trucks. He has a policy of replacing a tire when it is worn completely. This costs him Rs. 6,000 per tire. He has been advised that the replacement cost can be reduced to Rs. 4,500 if he replaces tires periodically. The past data has revealed the following replacement schedule

Table 8.22

Months after replacement	1	2	3	4	5	6
Proportion of the tires worn-out	5	12	20	45	75	100

What should be the optimal replacement policy?

6. A pipeline is due for repairs. It will cost Rs. 1,00,000 for repairs, which will last for 3 years. An alternative is to lay a new pipeline, which can work for 10 years. If the cost of money is 10% and there is no salvage value, what should be the decision?

7. Let *p* (*t*) be the probability that a machine in a group of 30 machines would breakdown in period *t*. The cost of repairing a broken machine is Rs. 200. Preventive maintenance ensures servicing of all the machines at the end of *T* units of time at a cost of Rs. 15 per machine. Find the optimum *T* which would minimize the expected total cost per period of servicing given that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots 10 \\ 0.13 & \text{for } t > 10 \end{cases}$$

8. A research team is required to attain a stable level of 50 members. The service schedule of the members is given below

Table 8.23

Year	1	2	3	4	5	6	7	8	9	10
Percent resignations at the end of the year	5	36	56	63	68	73	79	87	97	100

What should be the number of recruitments per year to maintain the required strength? If the promotion requires at least 8 years of service, what is the average length of the service after which a new recruit can expect promotion?

9. The management of a large hotel is considering the periodic replacement of the light bulbs fitted in its rooms. There are 500 rooms in the hotel and each rook has 6 bulbs. The current policy is to replace the bulbs as and when they fail. Per unit cost of replacing failed bulbs is Rs. 10.00. The new policy can cut the costs to up to Rs. 6.00. The past data reveals the following information:

Table 8.24

Months of use	1	2	3	4	5
Percent of bulbs failing by the end of the month	10	25	50	80	100

What should be the optimal replacement policy?

10. If the cost of capital is 10% per annum, which of the following investment plans should be opted

Table 8.25

Details	Plan A	Plan B
Initial cost (Rs.)	2,00,000	2,50,000
Salvage value after 15 years (Rs.)	1,50,000	1,80,000
Annual profit (Rs.)	25,000	30,000