Chapter 7

Game Theory- Extensive Form Games

Key words: Strategy, extensive form games, dominant strategy, Nash equilibrium, strategy profile, perfect information game, imperfect information game, sub game perfect Nash equilibrium.

Suggested readings:

1. Watson J., (2002), **Strategy: An Introduction To Game Theory**, W.W. Nortan & Company
2. Osborne M.J., (2001) **An Introduction to Game Theory**, Oxford University Press.

7.1 Introduction

Consider the following market situation.

A company, say, A Ltd. is holding a large share of the market and its product is very well known in the market. A second company, say, B Ltd., which is new, wishes to enter the market and is having an aggressive marketing policy. Since, with an aggressive marketing policy, B Ltd. can make a cut into the market share of A, its moves are being closely watched by A. In marketing promotions, advertising is a well known strategy and even a monopolist may have to resort to advertising in order to sell his product. If B does not enter into the market, then the annual profits of A are of tune Rs. 50 crores if it does not advertise its product and the profits move up to Rs. 65 crores if it advertises.

If company B decides to enter, the things are different. However, none of the companies is aware of the marketing plans of the other and they have to devise their marketing promotion plans independently and simultaneously. If both the companies choose to advertise their products, then profits for A would be of order Rs. 40 crores and for B, it would be of order of Rs. 25 crores. If both of them choose not to advertise, they both would get Rs. 20 crores each. If A chooses to advertise and B not, the profits would be Rs. 45 crores and Rs. 10 crores respectively, and if B chooses to advertise and A not, they both would get Rs. 30 crores each.

With this information, the two companies wish to decide their courses of action.

In this case, multiple decisions have to taken. First of all, B has to decide whether or not it wants to enter. If it does not enter, then A would have to decide whether it wants to advertise its product or not. If B enters, then they both have to decide independently whether their products are to be advertised or not.

Consider the following pictorial representation of the situation:

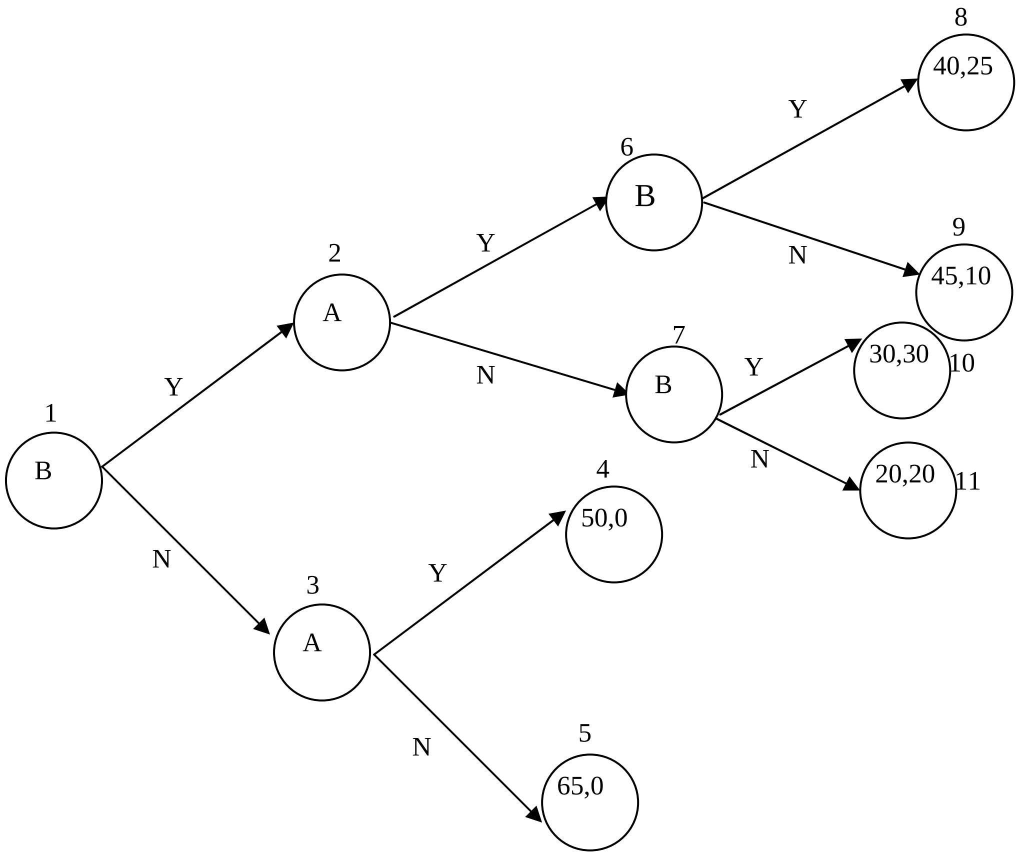

Fig 7.1

This presentation of a game is called **extensive form of the game**.

It can be seen that the above game involves a series of decisions on the behalf of two players before it could be concluded. Although normal or matrix form presentation of the above game is also possible, this form of presentation is more comprehensive.

7.2 Representing an extensive form game

Extensive form presentation of a game is presenting the game in form of a tree. The decision points in the game are represented by what we call as **nodes**. The actions arising from theses decisions are represented by **arrows**. In general a game in extensive form looks as follows:

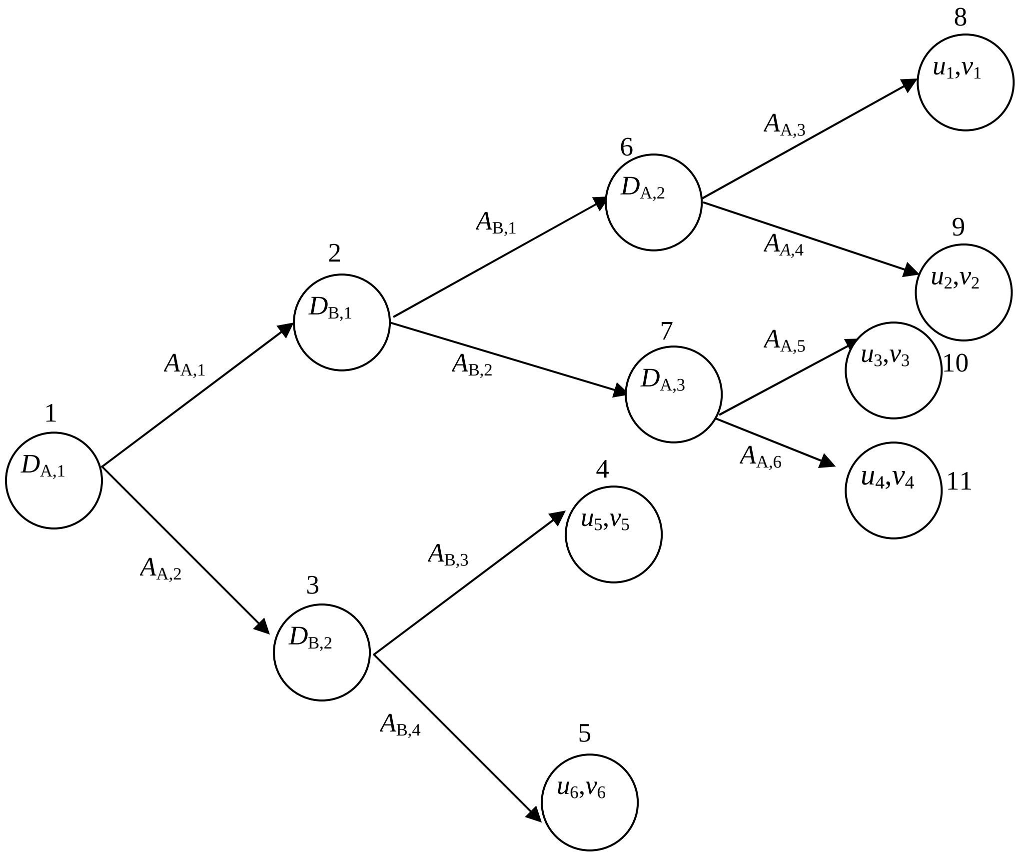

Fig. 7.2

Suppose that the game is being played by two players A and B, with the player a starting the game. A takes a decision $D_{A,1}$ at the node 1. This node at the beginning of the game is called the **initial node**. ***Every extensive form game has only one initial node***. Corresponding to his decision $D_{A,1}$, the player A can take two actions, viz. $A_{A,1}$ and $A_{A,2}$. Now, it is turn of player b to take the decision. Corresponding to action $A_{A,1}$ of player A, B takes decision $D_{B,1}$ at the node 2 and corresponding to action $A_{A,2}$ of player A, B takes decision $D_{B,2}$ at the node 3. If the player B has taken decision $D_{B,1}$, then his possible actions are $A_{B,1}$ and $A_{B,2}$. Similarly, corresponding to decision $D_{B,2}$, the possible actions are $A_{B,3}$ and $A_{B,4}$. In the later case, the game terminates at nodes 5 or 6, *u* being the (expected) pay-off or utility of player A and *v* being (expected) pay-off or utility of player B. As a convention, the pay-off of the player beginning the game is listed first. In case the player B has taken decision $D_{B,1}$, then player a again has to take a decision at node 6 or node 7. Then his actions would result in termination of the game at nodes 8,9,10 or 11 with the listed pay-offs. The pay-off nodes are called the **terminal nodes** and the results of actions of the players.

Now we can describe an extensive form game in general.

Every extensive form game starts with a single node (initial node or predecessor node). Branches originate from the predecessor node and point to (immediate) successor nodes. **If a node *b* is a successor to a node *a*, then the node *a* is a predecessor to the node *b*.** The end nodes are the terminal nodes. Initial node does not have any predecessor and terminal nodes do not have any successors. Although a successor node may have many predecessors but every node (except the initial node) has only one immediate predecessor node. The branches arising from the nodes represent the actions of the players. So it is possible to trace the sequence of decisions taken by any player. Any such sequence starting at the initial node and ending at the terminal node is called a **path**.

Branches of a tree cannot intersect with each other, i.e., the following tree is invalid

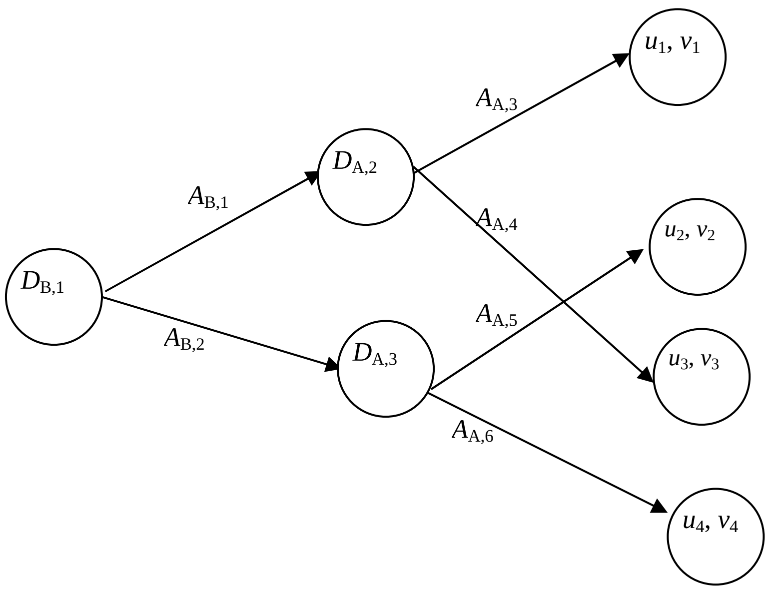

Fig 7.3

Information sets: The decision nodes are a part of what we call as information sets. An information set describes the information that player is having (about he strategies of the other players) when he has to take a decision. If the player does not know anything about the strategies chosen by the other players and he chooses his own strategy independently, then all his decision nodes arising from the actions of other player(s) will be joined by a dashed line. For example, in the example at the beginning, the decision taken by firm B whether to advertise or not is independent of the decision of firm A. So the two decision nodes of the decision taken by B will joined by a dashed line.

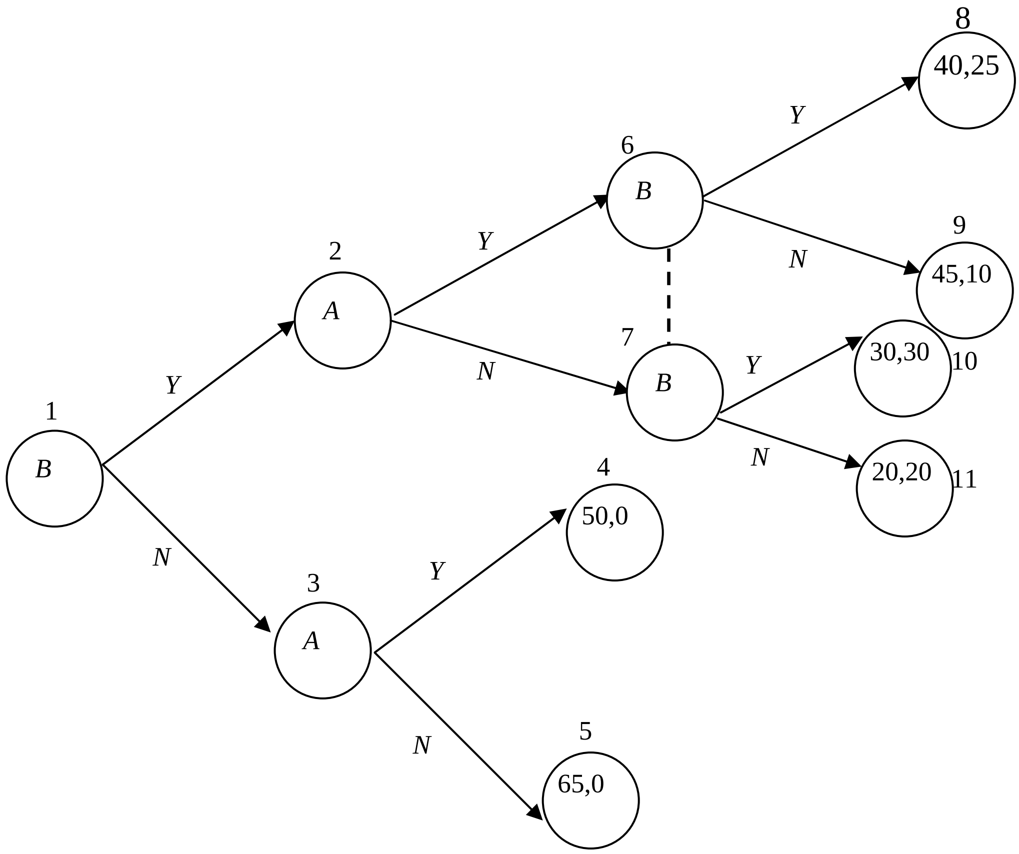

Fig 7.4

It should be noted that one information set contains information of one player at one point only. Thus the number of information sets is same as the number of decisions made in the game. In the above game the player A has to make decisions at nodes 2 and 3 whereas B has to make decisions at nodes 1, 6 and 7. Nodes 6 and 7 are in same information set as player B does not know about the decisions made by player A.

It should be noted that in any information set, all the nodes have same set of branches leading to the successor node. For example, the following set up is invalid.

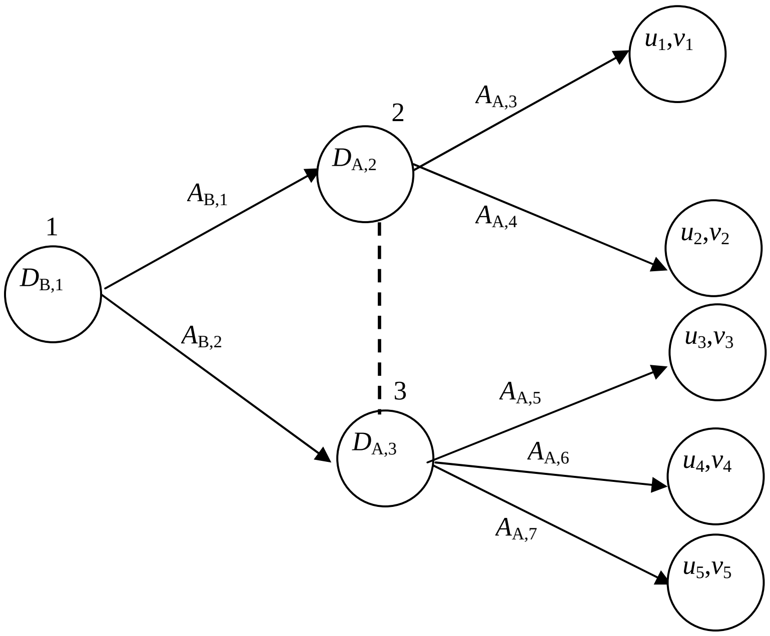

Fig 7.5

In this case, nodes 2 and 3 are not at the same level and before making his decision the player knows which node he is at. Thus 2 and 3 do not belong to the same information set.

Thus a game tree specifies the basic elements of a game, *viz,* (i) the number of players in the game; (ii) the set of possible actions of every player; (iii) the information which the players have when they are taking a decision; and (iv) the outcomes of actions of players.

Consider another example.

In large companies, professionals are generally hired by their Human Resource (HR) or personnel departments. However, sometimes, some senior officials hire the personnel directly (of course HR departments are intimated about new appointments). In such a situation, a senior official of R&D unit of a firm meets a researcher whom he wishes to hire for his company. There are three options before the official. He can hire the researcher directly, he may decide not to hire him or he can refer the case to HR unit of his company. However, if the case is referred to the HR unit, then it will be solely their decision to hire the researcher or not. In case of hiring the researcher by the official, HR unit will not interfere with his decision. In case, the researcher is hired, he will not be able to know whether he has been hire by the HR unit or was recommended by the official but he will have the option of either joining the company or refusing to join the company. The pay-offs of the game can be represented by numbers what we call as utilities. The utilities of various decisions are given below:

If the researcher is not hired (either by the official or by the HR unit after his case being sent to them), the pay-offs are all 0's. If the researcher is hired(either by the official or by the HR unit) and he does not join, then the official and the HR unit will get a pay-off of -1 each(there might be some penalty for exhausting company's resources unproductively) but the researcher gets a pay-off of 1(surely he has got a better option). If he has been hired by the official and he joins, then the researcher and the official get a pay-off of 1 each but the HR unit does not get anything. In case, he has been hired by the HR unit and he joins, then the researcher and the HR unit official get a pay-off of 1 each but the official gets a pay-off of ½ (after all it was he, who had found a potential researcher for the company). We want to present the game with the help of a game tree.

Let the official, being the first player of the game is represented by 1, the HR unit by 2 and the researcher by 3.

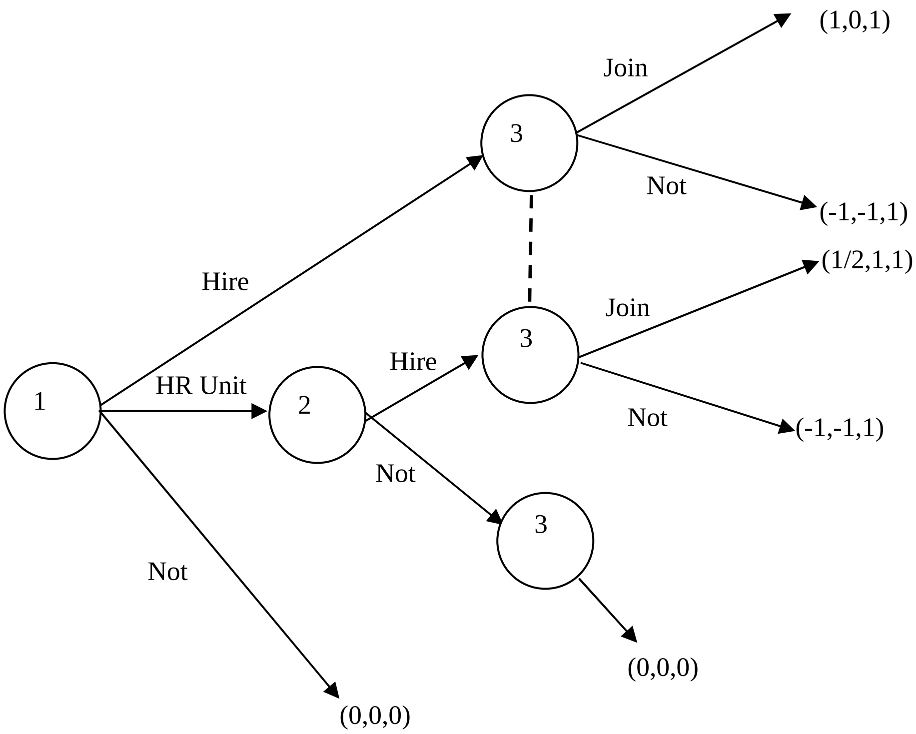

Fig 7.6

In this case, the first player has three options {hire, HR unit, not}, the second player has two options {hire, not}, and the third player has two options {join, not}.

7.3 Analysis of an extensive form game

In the earlier chapter, we have studied normal form of games, in which games have been described in terms of strategies. We may recall that a strategy is a comprehensive plan of action for a player which specifies all the possible actions that a player may take at any specific decision point. In the above case, the first player has strategy set {hire, HR unit, not} since he has to take decision first and his decision will not be influenced by the decision of the others. The second player has strategy set {(HR, hire), (HR, not)} since he may take decision only after first player decides to refer case to him, and the third player has strategy set {(HR, hire, join), (HR, hire, not), (hire, join), (hire, not)}. Each element of the strategy set is called a **strategy profile**. A strategy profile is a vector of strategies, one for each player. Thus the strategy profile (HR, hire, not) demonstrates that the official decides to refer the case to HR unit which in turn decides to hire the scientist who decides not to join the firm. Thus a strategy profile describes a path through the tree which is being followed and the set of all strategy profiles, in turn, describe the whole game tree. If pay-offs are given, then for each strategy profile, there is one and only one pay-off vector.

Perfect Information Game: A game is called perfect information game if at every decision point, a player is aware of the decisions made by his adversary. In other words, any information set has only one node and there are no two nodes joined by a dashed line.

Imperfect Information Game: If a player selects his moves without knowing the moves chosen by his adversary, then the game is called imperfect information game. In such games, one or more information sets may have nodes joined by dashed lines.

Solving an Extensive Form Game: In normal form games, we studied that a strategy is a comprehensive plan of action for a player. If the strategy is based on a probability distribution, i.e., at the time of making decision, a player is not aware of his adversary's decision, then the strategy is called a mixed strategy and if at the time of making a decision the player is aware of the decision made by his adversary, then the strategy chosen by him is called a pure or a regular strategy. In case of a pure strategy, the probability associated is 1.

In terms of pay-offs or utilities, a pure strategy *s* is called a dominant strategy if it has highest pay-off in class of all pure strategies. If for a strategy *t*, there exists another strategy *s* such that the pay-off of *s* is higher than the pay-off of *t*, then t is called a dominated strategy.

If the player is not aware of the decision made by his adversary, then he can form his best response.

Best Response: A mixed strategy (corresponding to a given probability distribution) is a best response of a player if it has then highest pay-off in class of all mixed strategies (corresponding to a given probability distribution).

Consider the following extensive form game:

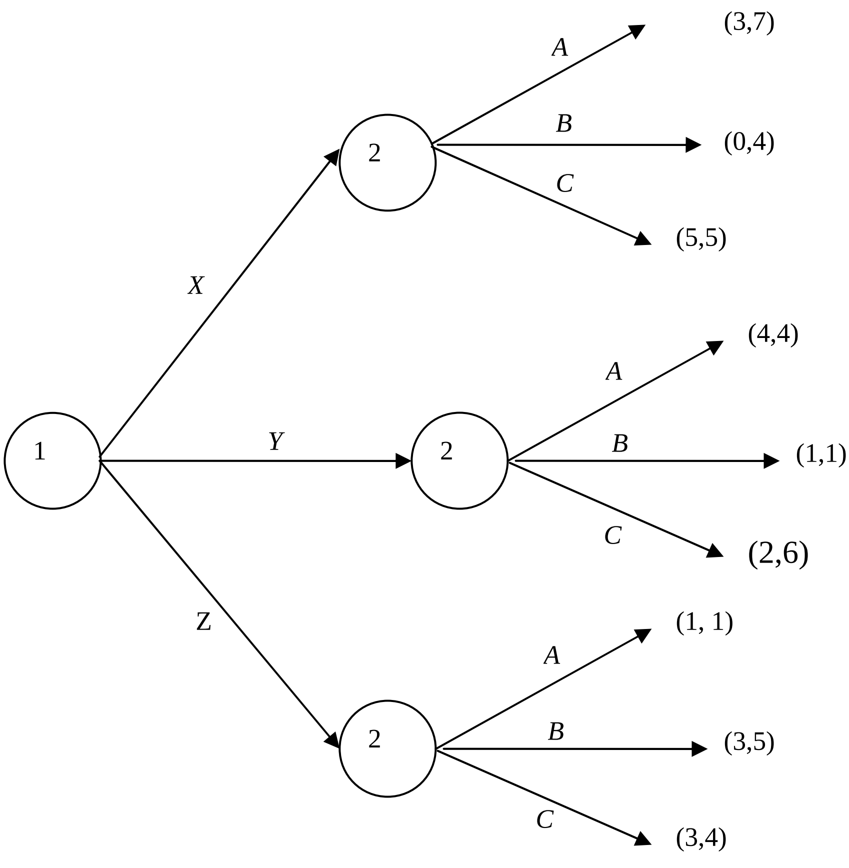

Fig. 7.7

For this game, we want to find the best response of player 1 for each of the strategies of player 2. We represent the game in matrix form as follows:

Table 7.1

2 / 1	*A*	*B*	*C*
X	(3,7)	(0,4)	(5,5)
Y	(4,4)	(1,1)	(2,6)
Z	(1,1)	(3,5)	(3,4)

Let player 1 believes that player 2 will, play the strategies *A*, *B* and *C* with probabilities, $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$ respectively. Then the expected pay-offs of player 1 corresponding to different strategies *X*, *Y* and *Z* are given by

$$E(U_X) = 3\left(\frac{1}{3}\right) + 0\left(\frac{1}{2}\right) + 5\left(\frac{1}{6}\right) = \frac{11}{6}$$

$$E(U_Y) = 4\left(\frac{1}{3}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{6}\right) = \frac{13}{6}$$

$$E(U_Z) = 1\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{6}\right) = \frac{14}{6}$$

Then the best response of player 1 is the one with pay-off $\frac{14}{6}$, i.e., the strategy *Z*.

Similarly, if player 2 attaches the probability distribution $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$ with the strategies of player 1, then his expected pay-offs are given as

$$E(U_A) = 7\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{19}{4}$$

$$E(U_B) = 4\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right) + 5\left(\frac{1}{4}\right) = \frac{17}{4}$$

$$E(U_C) \;=\; 5\left(\frac{1}{2}\right)+6\left(\frac{1}{4}\right)+4\left(\frac{1}{4}\right) \;=\; \frac{17}{4}$$

Then the best response of player 2 is strategy *A*.

Rationalizable strategies A rationalizable strategy is that strategy which a rational player would have a reason to choose from the set of available strategies for the information that he is having.

Rationalizability criterion strictly depends upon the information of the player. For example, consider the following game:

Table 7.2

1 \ 2	*A*	*B*	*C*
X	(4,4)	(1,6)	(1,5)
Y	(1,1)	(4,2)	(3,3)
Z	(0,0)	(3,1)	(3,2)

For player 2, strategy *A* is strictly dominated by strategy *B* so he will not choose this strategy. As a rational player, player 1 also knows that player 2 will not choose this strategy and thus this can be eliminated

Table 7.3

1 \ 2	*A*	*B*	*C*
X	(4,4)	(1,6)	(1,5)
Y	(1,1)	(4,2)	(3,3)
Z	(0,0)	(3,1)	(3,2)

.Now, for player 1, the strategy *X* becomes strictly dominated and he will not choose this strategy and this information is known to player 2 also. As a result, this strategy can also be eliminated.

Table 7.4

1 \ 2	*A*	*B*	*C*
X	~~(4,4)~~	~~(1,6)~~	~~(1,5)~~
Y	~~(1,1)~~	(4,2)	(3,3)
Z	~~(0,0)~~	(3,1)	(3,2)

Proceeding in the similar manner, we get

Table 7.5

1 \ 2	*A*	*B*	*C*
X	~~(4,4)~~	~~(1,6)~~	~~(1,5)~~
Y	~~(1,1)~~	~~(4,2)~~	**(3,3)**
Z	~~(0,0)~~	~~(3,1)~~	~~(3,2)~~

As rational players, 1 and 2 both know that the best course of action is to play with the strategy profile (*Y*, *C*).

7.4 Nash equilibrium

Named after strategist John Nash, Nash equilibrium refers to a situation when for each player, the expected or prescribed strategy is a best response to the other players' strategies. In mathematical terms a strategy profile $s = (s_1, s_2, ...s_i,...) \in S$, where S is the set of all strategy profiles, is a Nash equilibrium iff

$$s_i \in BR_i(s_1, s_2, ...s_{i-1}, s_{i+1},...) \quad \forall\ i$$

in the sense that

$$u_i(s_i, s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots) \ \geq \ u_i(s'_i, s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots) \forall\ s'_i \text{ and } s_i;$$

$$s'_i \text{ and } s_i \ \in S_i(s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots)$$

where $BR_i(s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots)$ is the best response set of the i^{th} player; $S_i(s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots)$ is the set of his all possible strategies and $u_i(s_i, s_1,\ s_2,\ \ldots s_{i-1}, s_{i+1}, \ldots)$ is his pay-off when he chooses strategy s_i.

In other words, Nash equilibrium is a point where the utility or payoff of each of the players is maximized.

Consider the following game of tossing a coin being played by two players. If the two players get the identical faces of the coin, each player receives a pay-off of 1 and in case of different faces, they receive nothing. In extensive form the game is as follows:

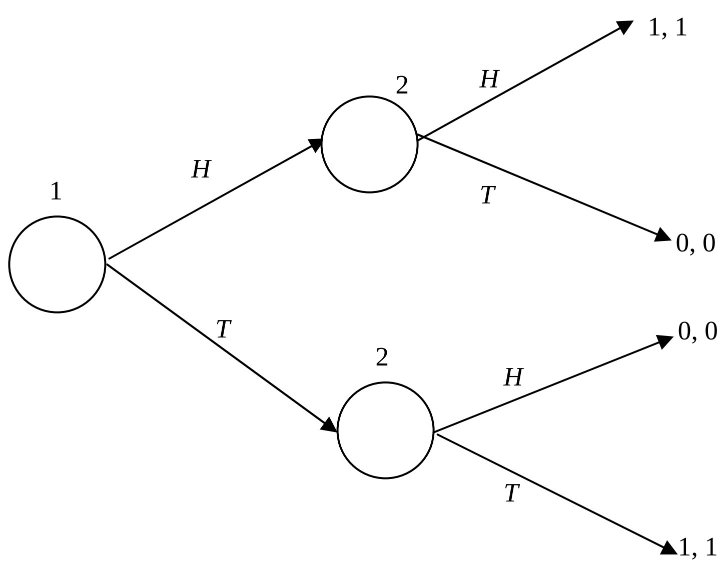

Fig 7.8

We will find if Nash equilibrium exist for this game. In matrix notation, the game can be represented as follows:

Table 7.6

1 \ 2	*H*	*T*
H	**(1,1)**	(0,0)
T	(0,0)	**(1,1)**

The strategy profiles {*H*, *H*} and {*T*, *T*} are the two Nash equilibria for this game.

Thus for some of the games, Nash equilibrium can be obtained by examining the pay-off or utility matrix. Nash equilibrium can also be obtained by elimination of dominated strategies. Consider the following game (also known as prisoners' dilemma):

Two criminal, captured by the authorities, have been known to commit a serious crime. But authorities do not have enough evidence to prosecute them for that crime. The evidences with the authorities can only help them to prosecute the criminals only for some minor offences. The only hope of authorities is to convince the criminals to admit the crime or to defect and depose against each other. If none of them admits their crime, they would be convicted for the minor offence which would fetch them a prison term of one year each. If any one of them defects and deposes against the other, he would be released but the other would get a prison term of three years. If both of them defect and depose against each other, they would get a prison term of two years each. Both the criminals have been kept into separate cells and hence they do not know about the decision made by each other. What should the criminals do?

We present the game in extensive form as follows:

If *D* stands for the strategy to defect and *N* stands for the strategy not to defect, then starting the game with first criminal, we have

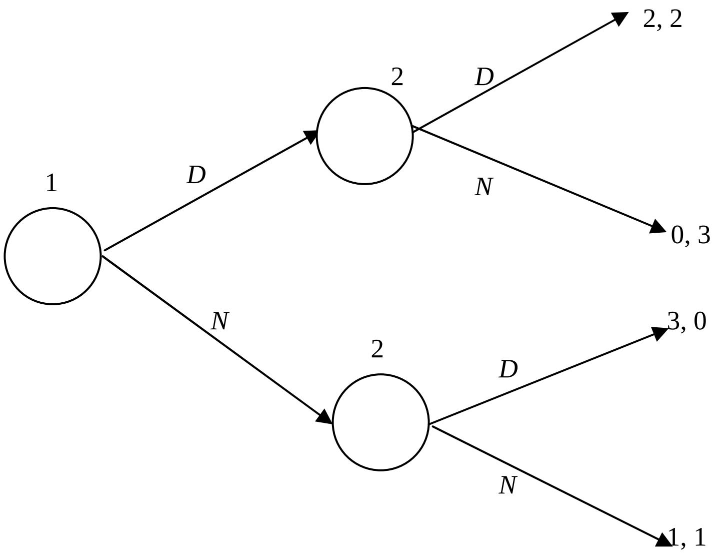

Fig .7.9

In matrix form, the game can be represented as follows:

Table 7.7

2 / 1	*D*	*N*
D	**(1,1)**	(0,3)
N	(3,0)	(2,2)

Obviously 1 will not choose not to defect as in that whatever will 2 do; 1 will get a higher prison term. So strategy *N* can be eliminated. In the reduced game, 2 will choose not to defect as this would ensure the minimum jail term for him (as well as for criminal 1)

Table 7.8

2 / 1	*D*	*N*
D	**(1,1)**	(0,3)
N	~~(3,0)~~	~~(2,2)~~

Thus the Nash equilibrium is the strategy profile {*D*, *D*}.

Another method of locating Nash equilibrium is to find the best responses of each of the players. Then a strategy profile in which all the strategies are the best responses will be a Nash equilibrium. Consider the following game:

Table 7.9

2 1	*A*	*B*	*C*	*D*
J	(4,3)	(3,5)	(1,4)	(2,5)
K	8,7)	3,4)	7,5)	4,6)
L	(7,3)	(4,1)	(9,8)	(0,4)
M	(3,8)	(7,8)	(3,4)	(5,6)

In this case, none of the strategies is a dominated strategy. Then we will first find out the best responses of player 1 to different strategies of player 2. For strategy *A* of player 2, the first column of the pay-off matrix suggests that strategy *K* is the best response. We underline the pay-off of the best response of player 1 for strategy *K*.

Table 7.10

2 1	*A*	*B*	*C*	*D*
J	4,3	3,5	1,4	**5**,5
K	**8**,7	3,4	7,5	4,6
L	7,3	4,1	**9**,8	0,4
M	3,8	**7**,8	3,4	**5**,6

Similarly we underline the best responses corresponding to strategies B, C and D. for strategy D we can see that we have two best responses.

Now we find out the best responses of player 2 for different strategies chosen by player 1.

Table 7.11

2 1	*A*	*B*	*C*	*D*
J	4,3	3,**5**	1,4	**5**,**5**
K	**8**,**7**	3,4	7,5	4,6
L	7,3	4,1	**9**,**8**	0,4
M	3,**8**	**7**,**8**	3,4	5,6

For strategy *J*, the best responses are strategies *B* and *D*; for strategy *K*, the best response is strategy *A*; for strategy *L*, the best response is strategy *C*; and for strategy *M*, the best responses are strategies *A* and *B* respectively. We notice that strategy profiles {*J*, *D*}, {*K*, *A*}, {*L*, *C*} and {*M*, *B*} maximize the pay-off of both the players. Hence these are the Nash euilibria for this game.

The strategy profile {*L*, *C*} is strict Nash equilibrium since this represents the best pay-off among all the strategies.

7.5 Solving the games with perfect information

The methods for solving an extensive form which we have discussed till now are based on the concept that the players choose their strategies independently and simultaneously. But this may not be the situation always and sometimes the decision making process is dynamic or sequential, i.e., a player will choose his move according to the move selected by the other player(s).

As an illustration recall the game of two companies A Ltd. and B Ltd. where A Ltd. is already there in the market and B Ltd. is planning to enter the market. The game starts when B has to take the decision of entering the market or not. If B enters the market, then only A has to decide whether to accommodate the competitor in the market or to uproot it from the market.

Besides market promotion, another time tested weapon of capturing the market is initiation of price war which has often been practiced by large companies to eliminate small companies. In order to uproot B, A may choose to initiate price war. Now, if B stays out of market, it does not gain or loose anything, but A being the sole player in the market, will be the net gainer. Suppose that pay-off of A in this situation is 5. If B enters and A chooses to adjust, both the companies will have a pay-off of 3 each. If B enters and A initiates a price war, both will loose and their payoffs will be -2 for A and -3 for B.

In this game, A's decision is subsequent to the decision of B. The game can be represented as

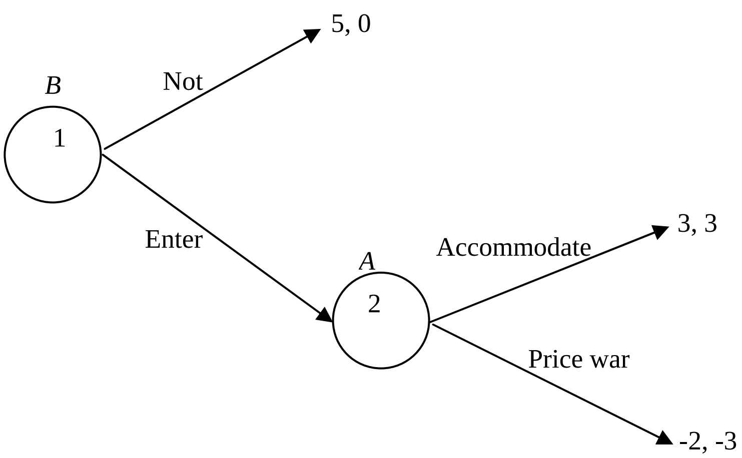

Fig 7.10

A Ltd. needs to take a decision only if B Ltd. decides to enter the market and it can not decide before hand whether to initiate a price war or not. In such dynamic situations, the players have to be rational whenever they have to take decision. Such rationality is called **sequential rationality**. We define sequential rationality formally as follows:

Sequential rationality Sequential rationality is a sequence of optimal decisions or an optimal strategy for a player which would maximize his expected pay-off conditional on every information set at which the player may have to take an action. Thus sequential rationality specifies an optimal action for those information sets also where he might not reach or does not believe to reach.

In the above case, $\{E, A\}$ will be an optimal strategy only if A Ltd. needs to take a decision.

In such sequential games, in order to obtain an optimal strategy, we start analyzing the tree from the end. At each information set, we evaluate the strategies and strike off the dominated strategies. The process continues backwards till an optimal; strategy is obtained. This process of analyzing the game from back to front is called the **backward induction**.

A strategy s_i for the player i, which is dominated by another strategy s'_i at some information set is called a **conditionally dominated** strategy. In backward induction, we eliminate the conditionally dominated strategies.

In the above example {*E*, *P*} is conditionally dominated by {*E*, *A*} and is eliminated. {*E*, *A*} is a Nash equilibrium. Another Nash equilibrium of this game is {*N*, *A*}**!**

It may be noted that {*N*, *A*} is also a Nash equilibrium as A, although threatening to initiate a price war, may not be in a position to do so. The company B also knows that. So there is a possibility that right now B is not entering the market but if at some time in future it decides to do so, it knows that only rational behavior of A would be to accommodate it in the market. Thus the game in matrix form may be represented as

Table 7.12

A \ B	*A*	*P*
E	**(3,3)**	(-2,-3)
N	(0,5)	(0,5)

Consider the following game in extensive form which we will solve by backward induction.

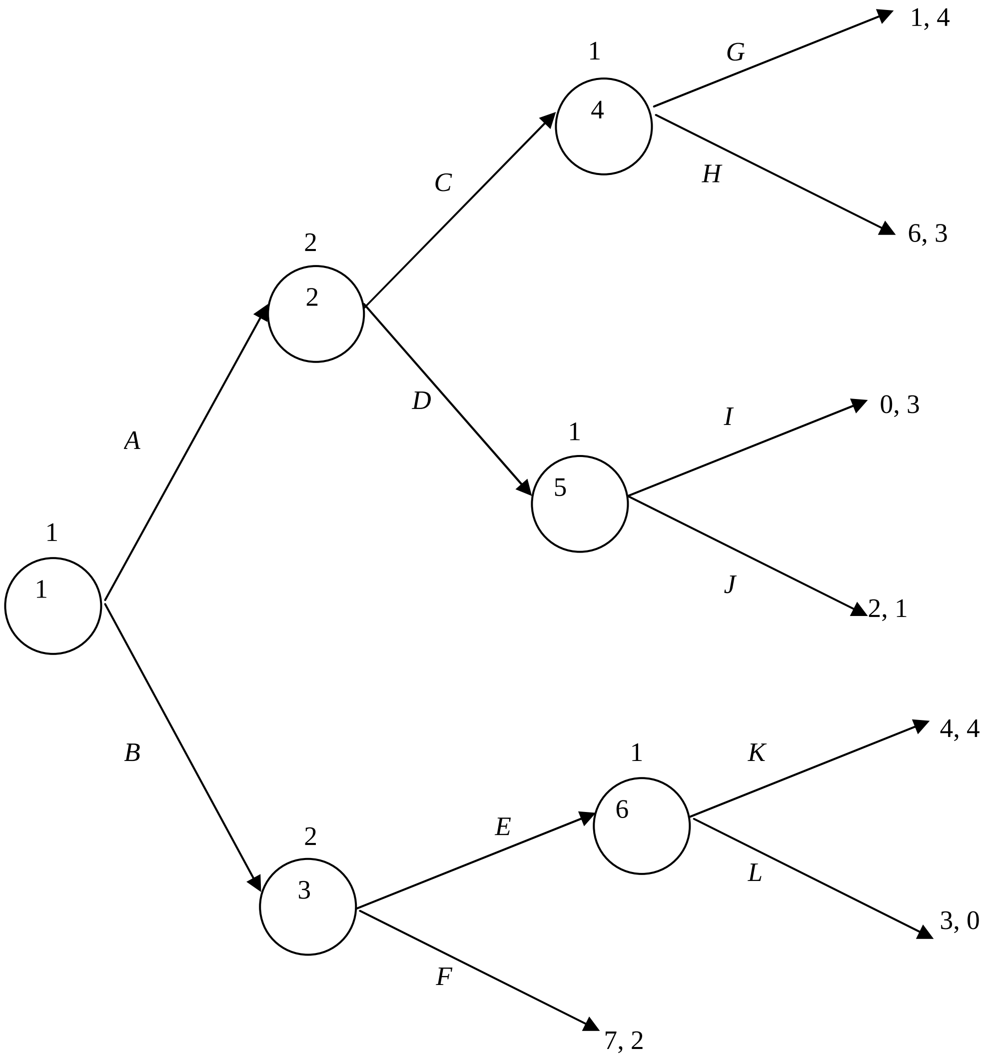

Fig. 7.11

There are four nodes at which player 1 has to make decisions and two nodes at which player 2 have to make decisions.

At node 6, 1 has to decide between actions *K* and *L*. Obviously he will choose *K* and as a result we can eliminate *L*. We move backwards and reach decision node 3 at which 2 has to decide between *E* and *F*. If he chooses *F*, them his pay-off would be 2. If he chooses *E* after which 1 would choose *K*, his pay-off will be 4. Obviously 2 is not going for *F*.

At decision node 5, 1 will go for action *J* and at decision node 4, he will go for action *H*. At decision node 2, if player 2 chooses *C*, he will get a pay-off of 3(as 1 would go for *H*) but if he chooses *D*, his pay-off would be 1(as 1 would go for *J*). At this stage the optimal action for 2 is to go for *C*. At decision node 1, if 1 goes for *A*, he would get a pay-off of 6(as 2 would choose *C* and 1 will again go for *H*) but if he goes for *B*, his pay-off would be 4. So 1 will choose *A*.

Then the optimum strategy for two players is {*A*, *H*, *C*} and this is a Nash equilibrium.

Now suppose that we modify the game a little and has the following pay-offs

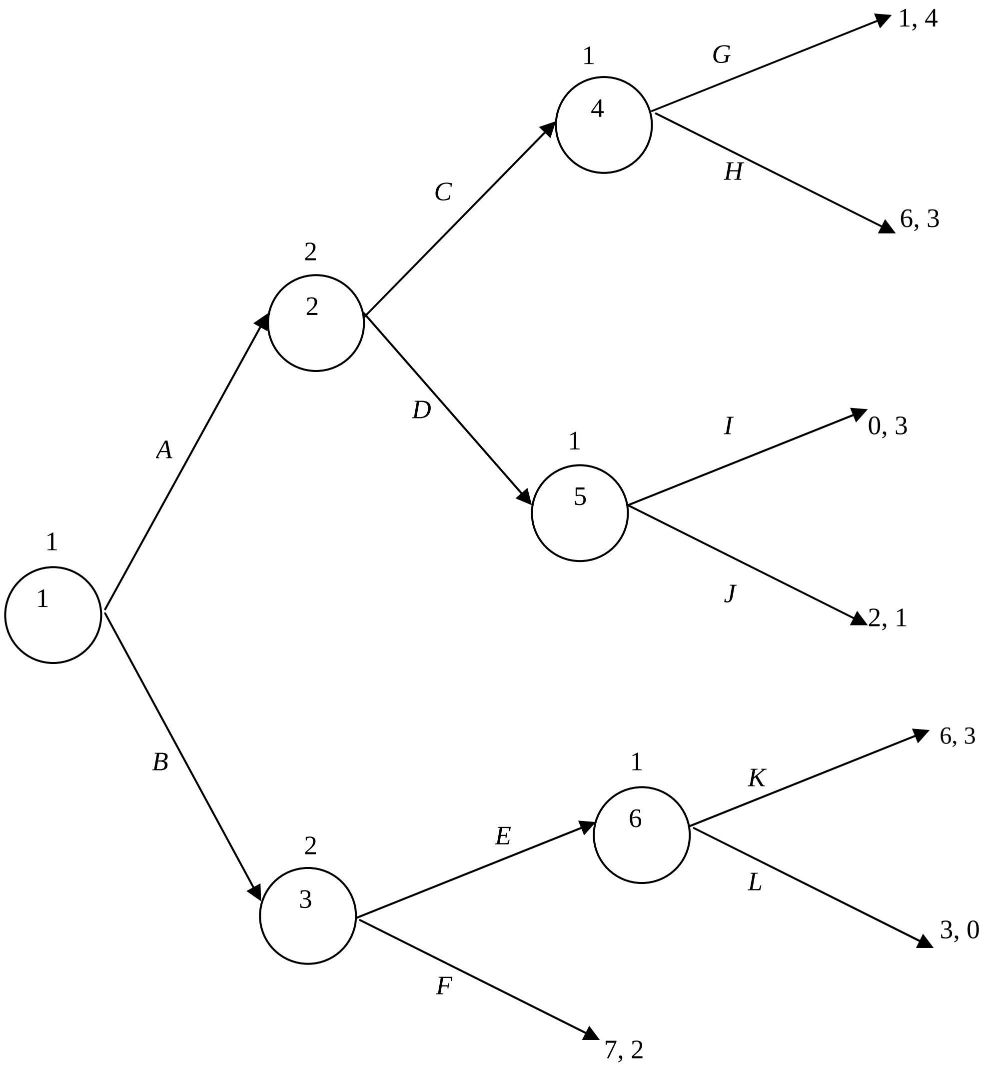

Fig. 7.12

Then what happens? Proceeding in the similar manner, we can see that now two Nash equilibria exist namely, {*A*, *H*, *C*} and {*B*, *K*, *E*}

7.6 Sub games of a game

A sub game of a game is a subset of information sets which is a complete game in itself. Recall the game considered above. At decision node 4, we have a complete game which would pay a pay-off of (1, 4) to the players 1 and 2 if 1 goes for G and a pay-off (6, 3) if he chooses H. so player 1 would go for *H*. Thus the strategy profile {*H*} is a Nash equilibrium for this game starting at node 4.

Similarly, at nodes 5 and 6 also we have complete games having Nash equilibria as {*I*} and {*K*] respectively. Again at nodes 2 and 3, we have sub games with Nash equilibria (*H*, *C*} and {*K*, *E*} respectively.

These games which start at a node other than the initial node are called proper sub games. Now we formally define a sub game.

Definition In an extensive form game, a node *x* and its successors constitute a sub game if they form a proper game in themselves (with the node *x* serving as the initial node)in the senses that they do not share an information set which contains some nodes other than *x* or its successors.

If we modify the game considered above as follows

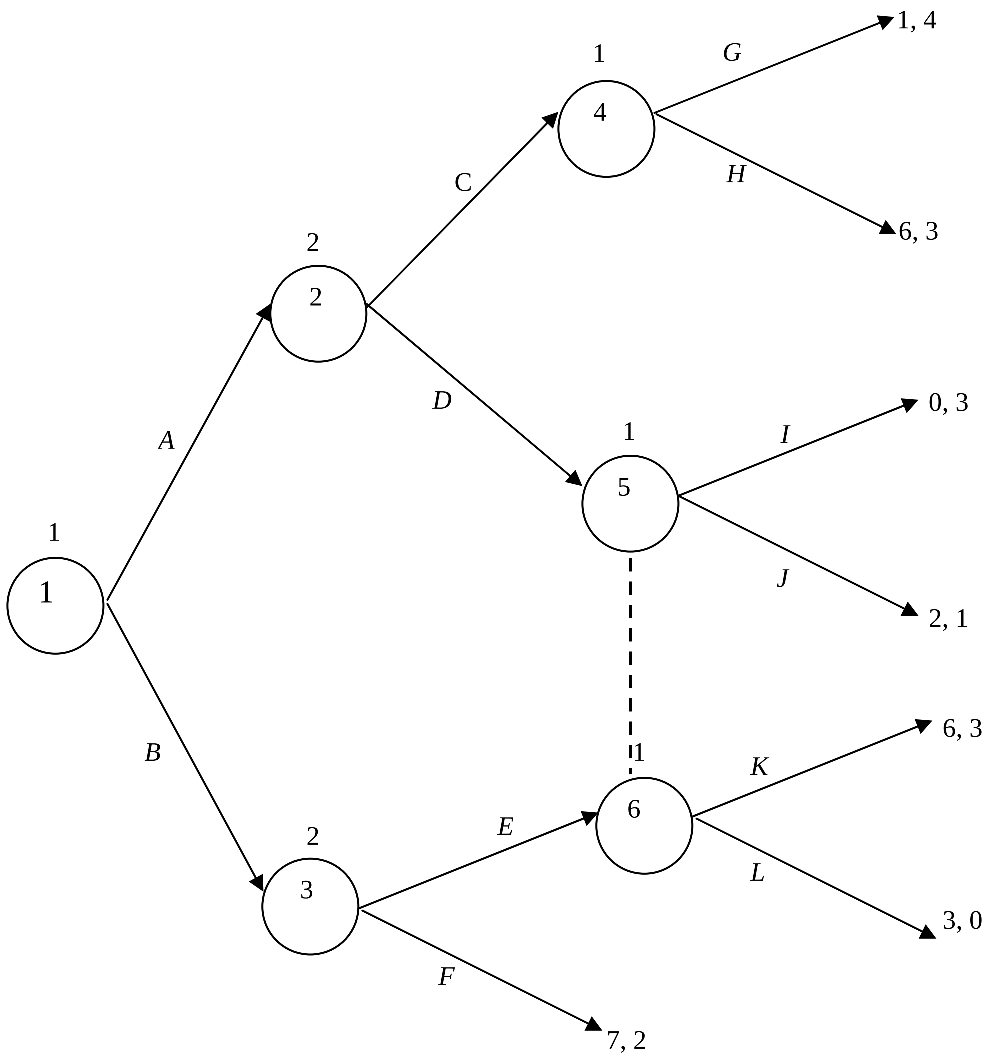

Fig. 7.13

Then nodes 2 and 3 do not initiate any sub game although at nodes 4, 5 and 6 we have sub games. However, consider the game

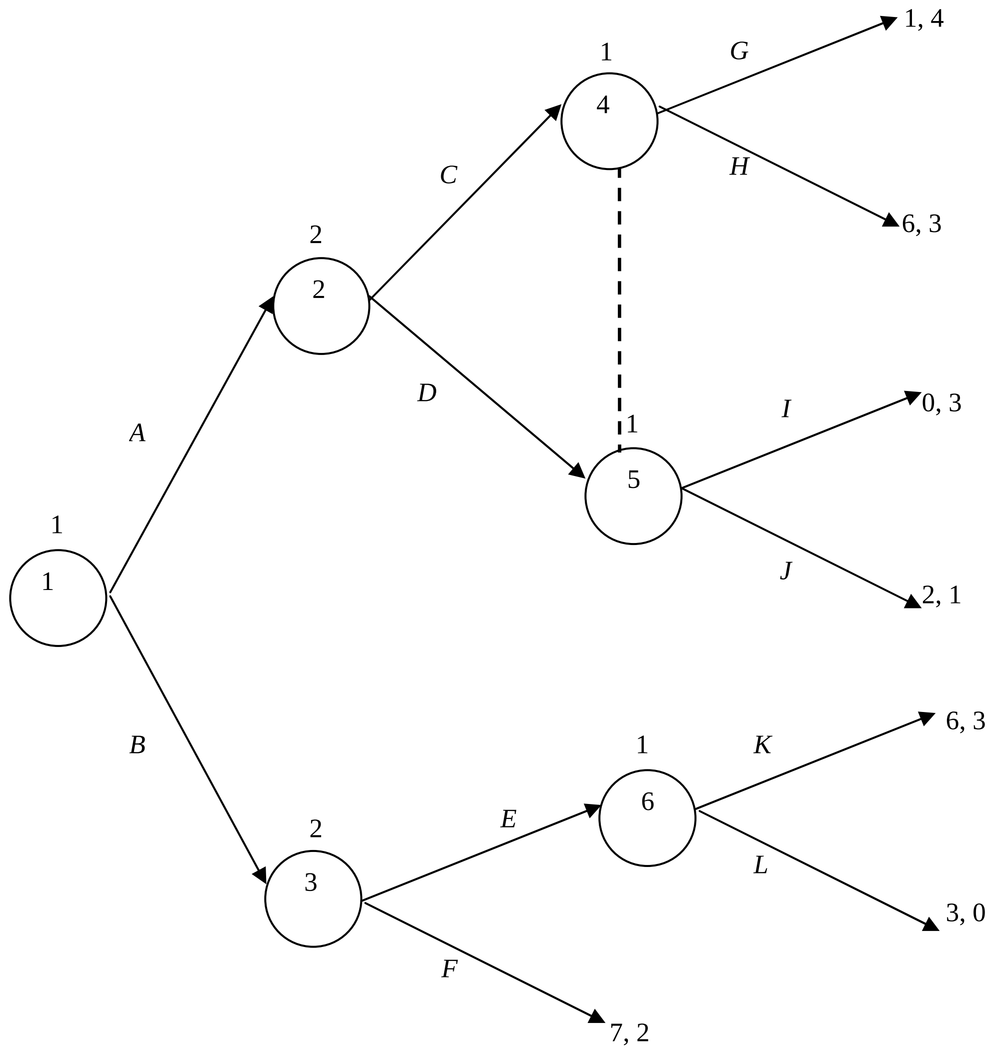

Fig. 7.14

This game has sub games at nodes 2 and 3 also.

Every game is a sub game of itself.

Sub game perfect Nash equilibrium A strategy profile is called a sub game perfect Nash equilibrium if it contains Nash equilibria for every sub game of the game. In other words a strategy *s* is a sub game perfect Nash equilibrium if for any sub game of the game, there exists a Nash equilibrium in *s*.

To understand the idea considers the following game:

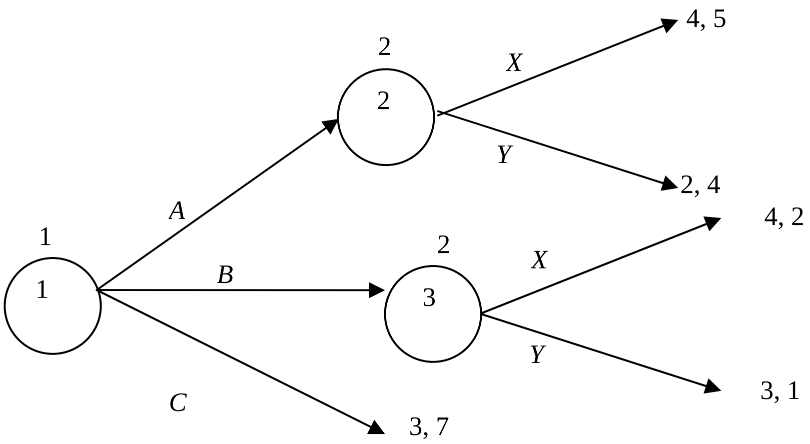

Fig 7.15

The matrix of the original game is

Table 7.13

2 / 1	*X*	*Y*
A	**(4,5)**	(2,4)
B	(4,2)	(3,1)
C	(3,7)	(3,7)

For this game Nash equilibrium is {*A*, *X*}.

Now consider the following sub game of the game (the only proper sub game)

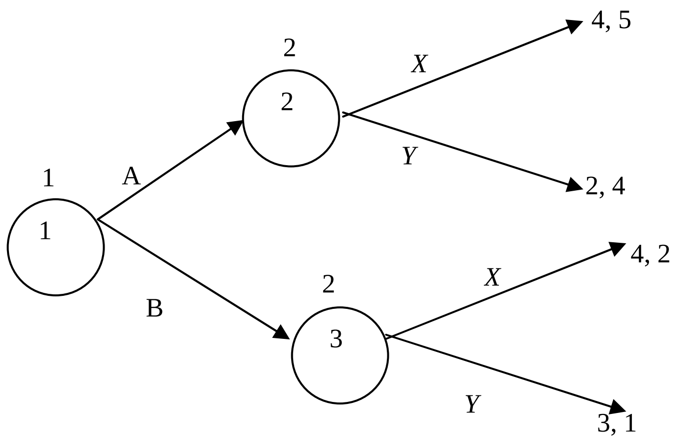

Fig. 7.16

For this sub game the matrix is

Table 7.14

2 / 1	*X*	*Y*
A	**(4,5)**	(2,4)
B	(4,2)	(3,1)

For this sub game Nash equilibrium is $\{A, X\}$. So $\{A, X\}$ is a sub game perfect Nash equilibrium of the original game.

We modify the game a little as follows:

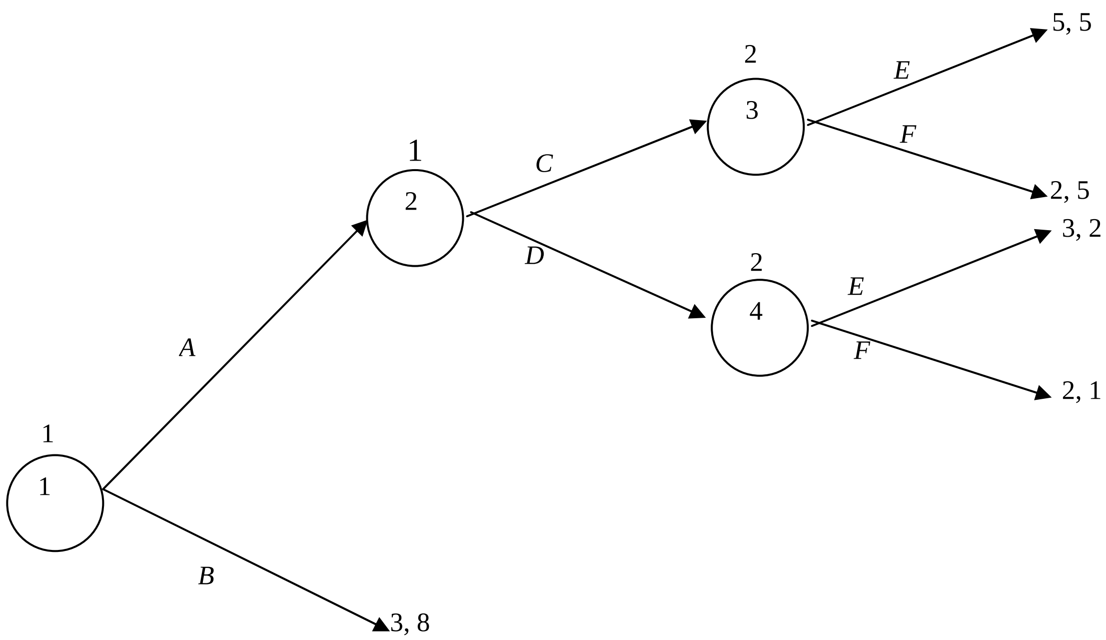

Fig 7.17

The entire game can be represented as

Table 7.15

2 / 1	*E*	*F*
AC	**(5,5)**	(2,5)
AD	(3,2)	(2,1)
BC	(3,8)	**(3,8)**
BD	(3,8)	**(3,8)**

The Nash equilibria are {*AC*, *E*}, {*BC*, *F*} and {*BD*, F}

Now consider the sub game of this game:

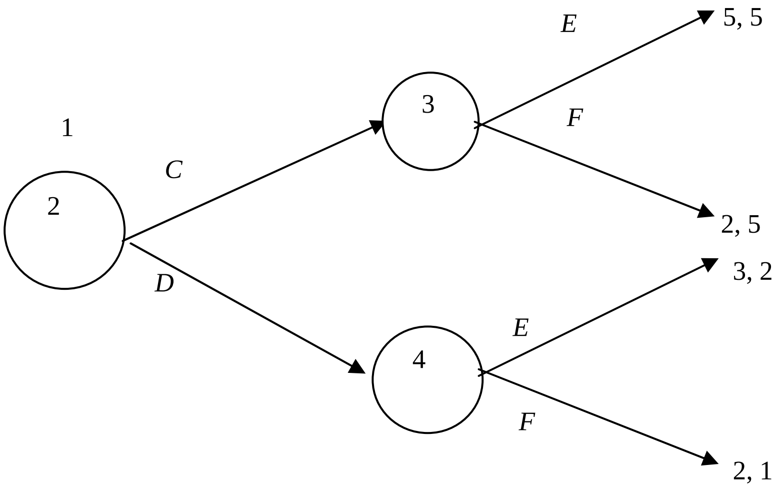

Fig. 7 18

The sub game matrix is

Table 7.16

2 / 1	*E*	*F*
C	**(5,5)**	(2,5)
D	(3,2)	(2,1)

In the proper sub game, there is only one Nash equilibrium, namely, {*C*, *E*}. So, although in the original game, there are three Nash equilibria namely {*AC*, *E*}, {*BC*, *F*} and {*BD*, *F*}, but there is only one sub game perfect Nash equilibrium namely {*AC*, *E*}.

Consider another game:

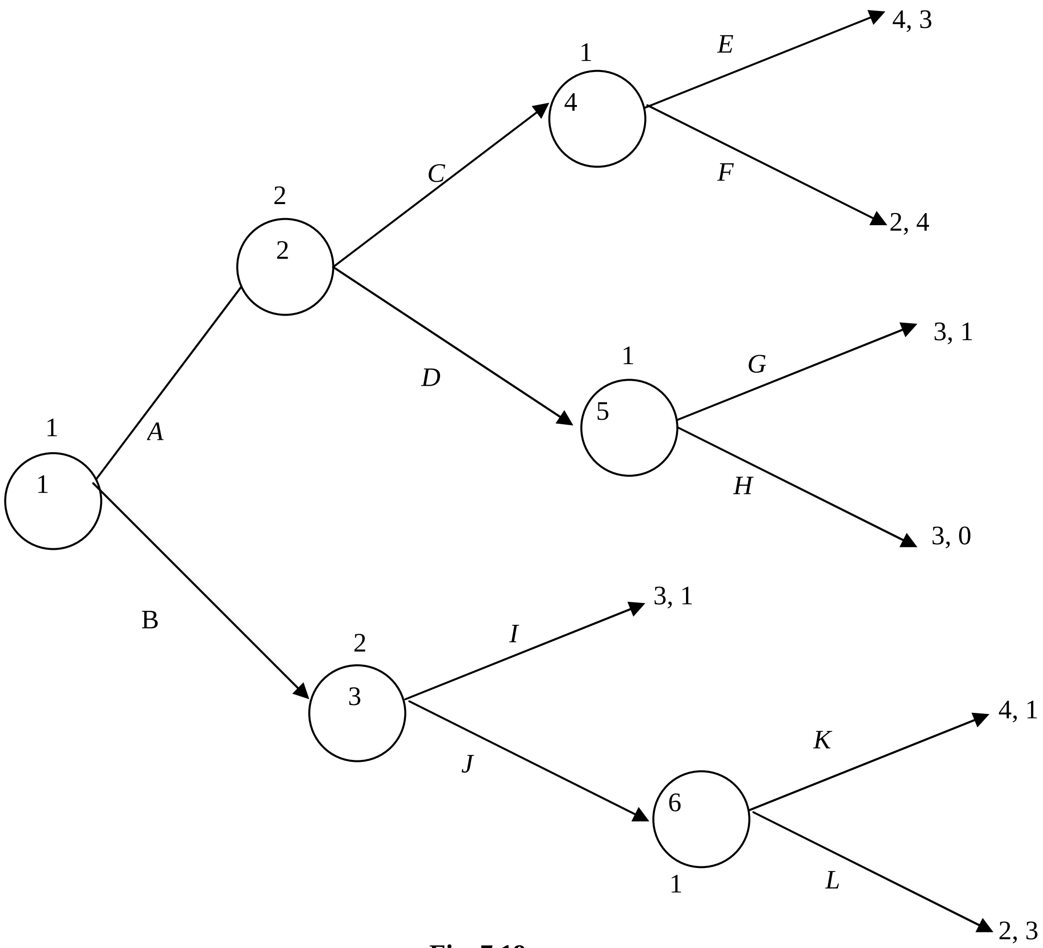

Fig. 7.19

First of all we solve the complete game by backward induction

Start with decision node 6. Optimal action for 1 is to choose *K* which he can choose if 2 goes for *J* when 1 has chosen *B* at decision node 1. Thus the optimal strategy profile becomes {*B*, *K*, *J*} with pay-off (4, 1). At decision node 5, 1 will go for G if 2 has gone for *D* at decision node 2. At decision node 4, 1 will go for *E* as 2 has gone for *C* at decision node 2. So optimal actions would be C and *E* if 1 goes for *A* at decision node 1 and the optimal strategy profile then would be {*AE*, *C*} with pay-off (4,3). Thus the game gas two Nash equilibria namely {*A*E, *C*} and {*BK*, *J*}.

Now consider the sub games starting at nodes 2 and 3. For these sub games {*E*, *C*} and {*K*, *J*} are Nash equilibria respectively which are contained in {*AE*, *C*} and {*BK*, *J*}. But none of these is a sub game perfect Nash equilibrium.

The matrix form representation of the entire game is

Table 7.17

1 \ 2	*C*	*D*	*I*	*J*
AE	**(4,3)**	(4,3)	(4,3)	(4,3)
AF	(2,4)	(2,4)	(2,4)	(2,4)
AG	(3,1)	(3,1)	(3,1)	(3,1)
AH	(3,0)	(3,0)	(3,0)	(3,0)
BK	(4,1)	(4,1)	(3,1)	**(4,1)**
BL	(2,3)	(2,3)	(3,1)	(2,3)

For the sub games starting at nodes 2 and 3, the matrices are

Table 7.18

1 \ 2	*C*	*D*
E	**(4,3)**	(4,3)
F	(2,4)	(2,4)
G	(3,1)	(3,1)
H	(3,0)	(3,0)

And

Table 7.19

2 / 1	*I*	*J*
K	(3,1)	**(4,1)**
L	(3,1)	(2,3)

In the first sub game, Nash equilibrium is occurring at {*E*, *C*} and in the second sub game it is occurring at {K, J}. So for the original game, Nash equilibria correspond to {*AE*, *C*} and {*BK*, *J*} which are also sub game perfect Nash equilibria.

In games with the perfect information backward induction yields equilibria which are sub game perfect.

Problems

1. Represent the following games in extensive form:

(a) A firm A is a well established firm in an industry and another firm B wants to enter the market. Firm A is watching the moves made by firm B. If B enters the market there is going to be tough competition and the two firms may engage in market promotion or price war or both. If B does not enter, then A will be the sole player of the market with a pay-off of 10 whereas B will not gain or loose anything. The following table represents their pay-offs if they decide to compete.

Table 7.20

B / A	Market promotion	Price war	Both
Market promotion	(4,6)	(6,6)	(1,-2)
Price war	(0,0)	(-2,-2)	(-4,-5)
Both	(-3,-4)	(-5,-5)	(-8,-8)

(b) In the above case another possibility is that either one or both will try to adjust with the other in the market. In that case the following table represents their pay-offs

Table 7.21

A \ B		Adjust	compete		
			Market promotion	Price war	Both
Adjust		(12, 10)	(7,8)	(1,1)	(-1, -1)
compete	Market promotion	(8,7)	(4,6)	(6,6)	(1,-2)
	Price war	(1,-2)	(0,0)	(-2,-2)	(-4,-5)
	Both	(-3,-3)	(-3,-4)	(-5,-5)	(-8,-8)

(c) Consider the following normal form representations of the two-player games.

Table 7.22

1 \ 2	*X*	*Y*
K	(5,3)	(3,1)
L	(6,2)	(7,0)

Table 7.23

1 \ 2	*X*	*Y*	*Z*
K	(0,4)	(2,4)	(8,4)
L	(7,3)	(2,7)	(2,3)
M	(5,6)	(2,4)	(9,5)

(d) In a play school game, the teacher chooses two children who have to choose words simultaneously written on a board in front of them. There are three words namely, apple, mango and banana. If both the children pick up the same word, they will get one chocolate each. If they pick up different words, one of them will be a winner and will get one chocolate whereas the other will get nothing. The winning rule is that of apple over mango, mango over banana; and banana over apple.

2. Represent the following extensive form games in normal form:

(a)

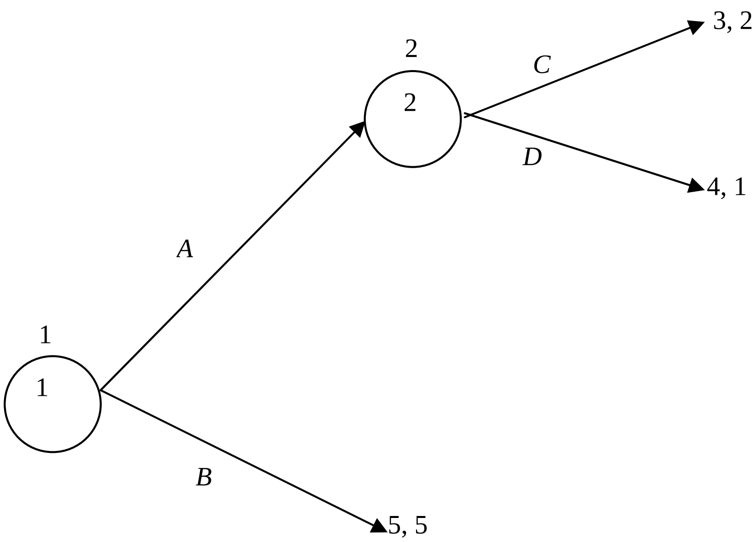

Fig. 7.20

(b)

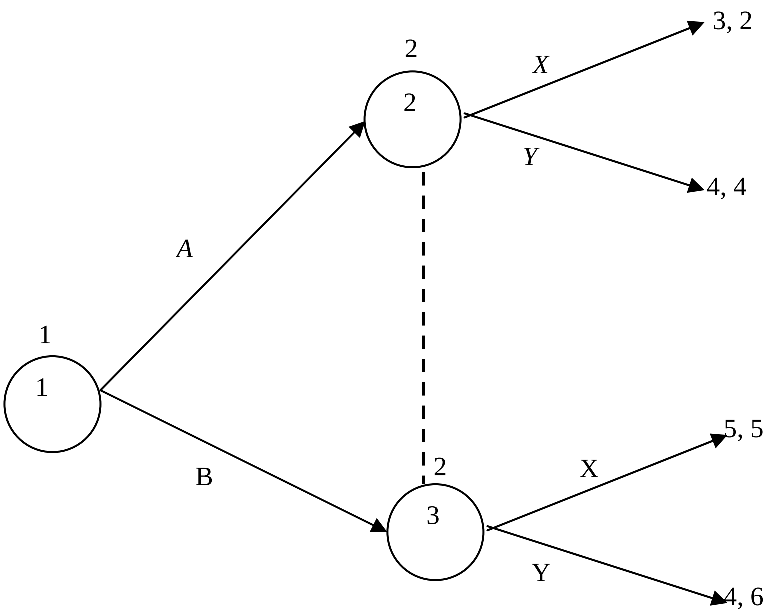

Fig. 7.21

(c)

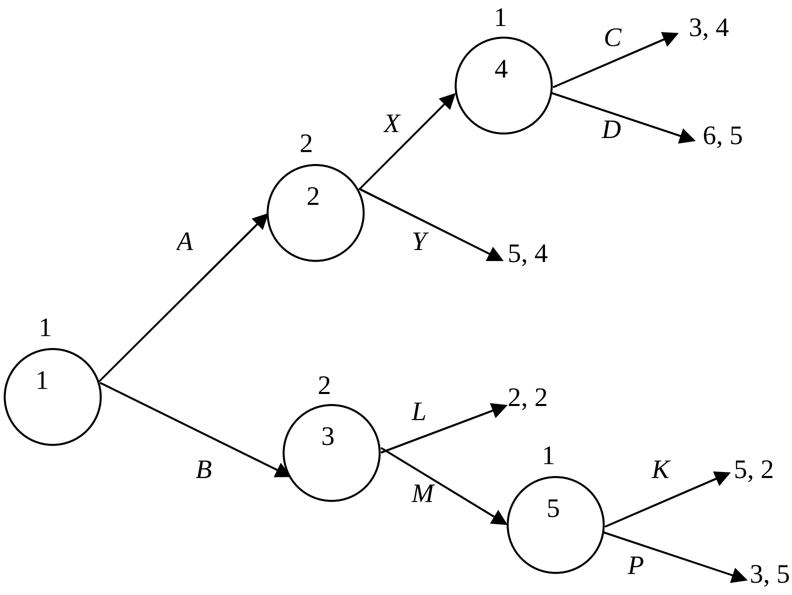

Fig. 7.22

(d)

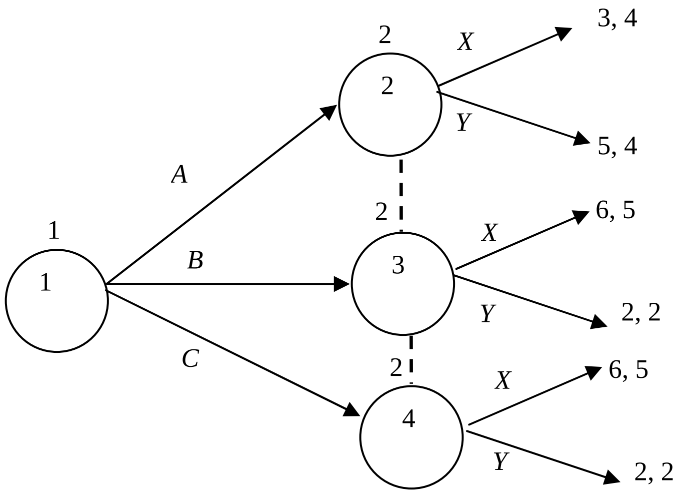

Fig. 7.23

3. For the following games, determine the best response strategies:

(a)

Table 7.24

2 / 1	*X*	*Y*	*Z*
K	(5,0)	(0,6)	(3,3)
L	(1,5)	(5,1)	(7,4)
M	(2,3)	(4,5)	(6,5)

(b)

Table 7.25

2 / 1	*X*	*Y*	*Z*
K	(6,0)	(2,3)	(5,4)
L	(4,3)	(2,10)	(2,2)
M	(3,9)	(2,3)	(9,5)

When the corresponding probability distributions are

(i) $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (ii) $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$

(iii) $\left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$ (iv) $\left(0, \frac{1}{2}, \frac{1}{2}\right)$

(v) $\left(\frac{1}{3}, \frac{5}{12}, \frac{1}{4}\right)$ (vi) $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

4. Represent the following games in extensive form, normal form and find the Nash equilibrium of the game, if it exists:

(i) Two players 1 and 2 simultaneously select numbers between 0 and 1, say x and y. If $x > y$, 1 gets a pay-off of $(x+ y)$ /2 and 2 gets a pay-off of $(x- y)$ /2. If $x < y$, 1 gets a pay-off of $(y- x)$ /2 and 2 gets a pay-off of $(x+ y)$ /2. If $x = y$, both get a pay-off of ½ each.

(ii) Suppose that a third player 3 is also there in the game that will play the game with the winner in part (i). He will choose a number *z*. If $z > (x+y)/2$, then he will get a pay-off of $\left(\frac{z+\frac{x+y}{2}}{2}\right)$ and the other player will get a pay-off of $\left(1-\frac{z+\frac{x+y}{2}}{2}\right)$. If $z = (x+y)/2$, both get a pay-off of ½ each.

5. Find the Nash equilibrium of the following game:

Table 7.26

1 \ 2	*X*	*Y*	*Z*
K	(9,5)	(5,9)	(4,1)
L	(5,9)	(9,5)	(1,1)
M	(1,1)	(1,1)	(3,3)

6. Solve the following games by backward induction:

(i)

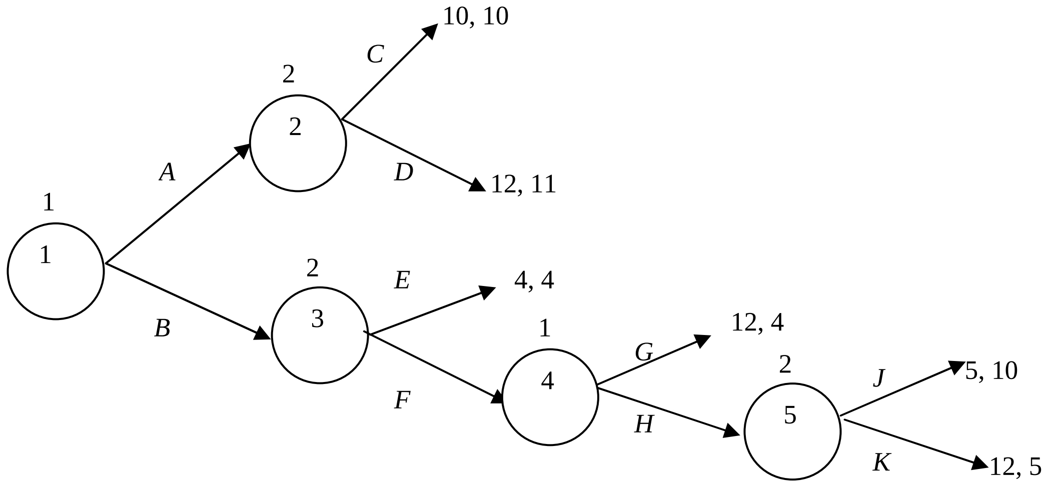

Fig. 7.24

(ii)

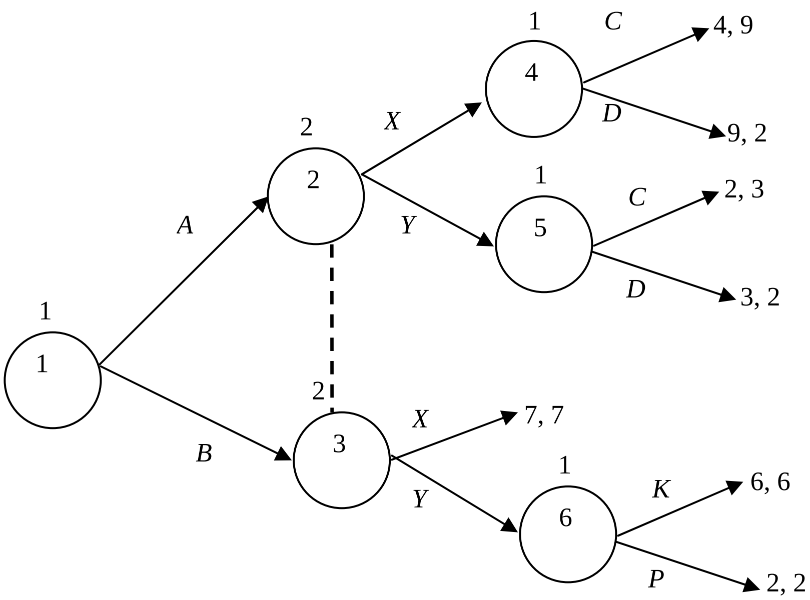

Fig. 7.25

(iii)

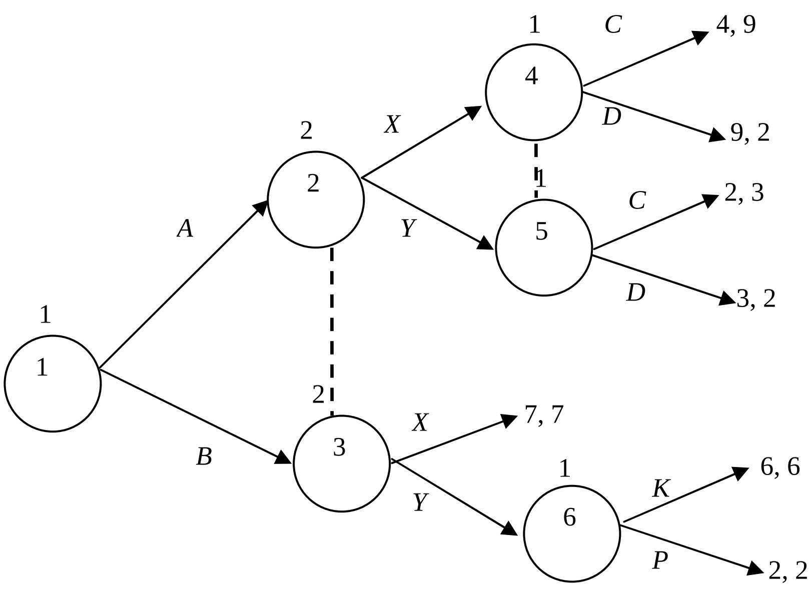

Fig. 7.26

(iv)

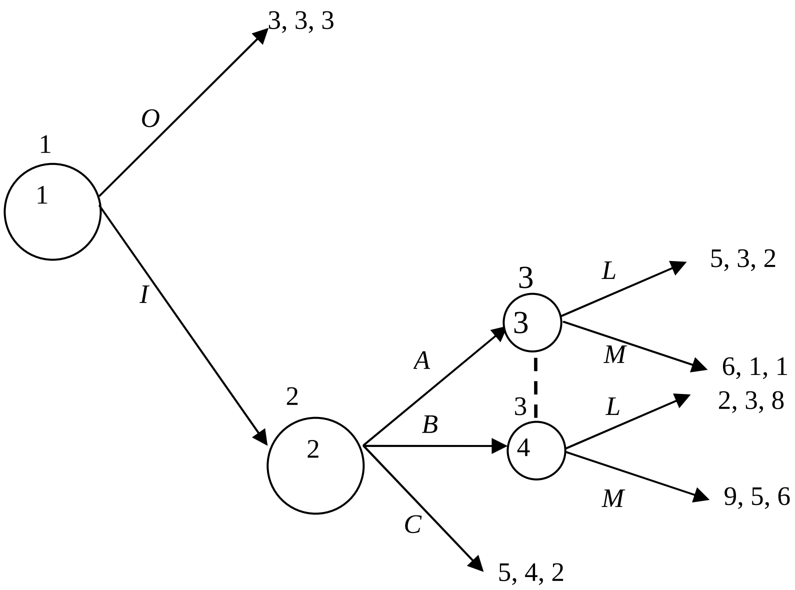

Fig. 7.27

7. For the games in problems 1 and 2, find the rationalizable strategies.

8. Find the Nash equilibrium for each of the games in problem2.

9. For the following games, find if Nash equilibria (if exist) are sub game perfect Nash equilibriua also.

(i)

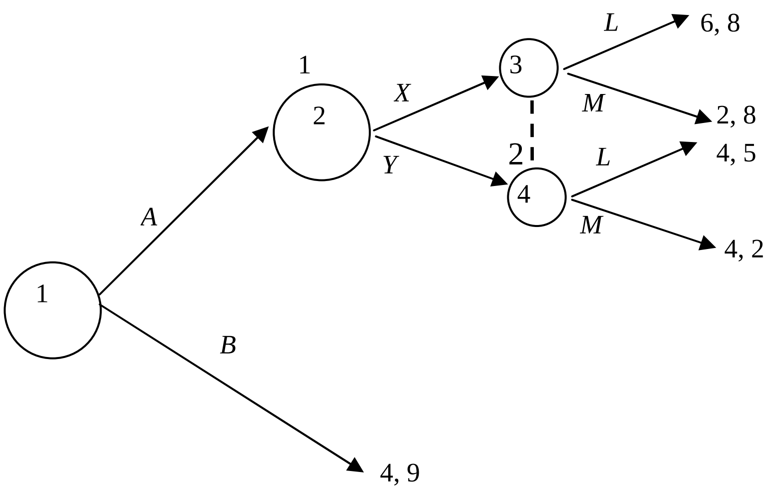

Fig. 7.28

(ii)

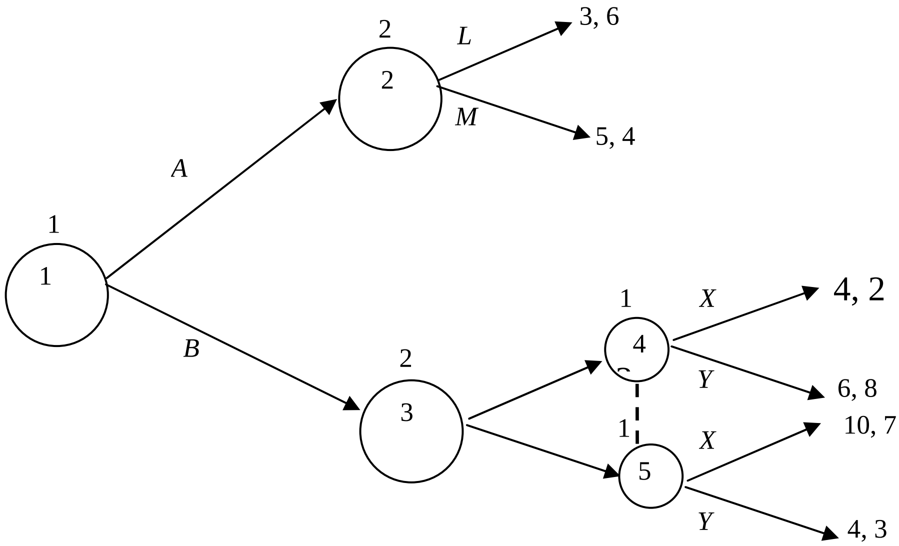

Fig. 7.29

(iii)

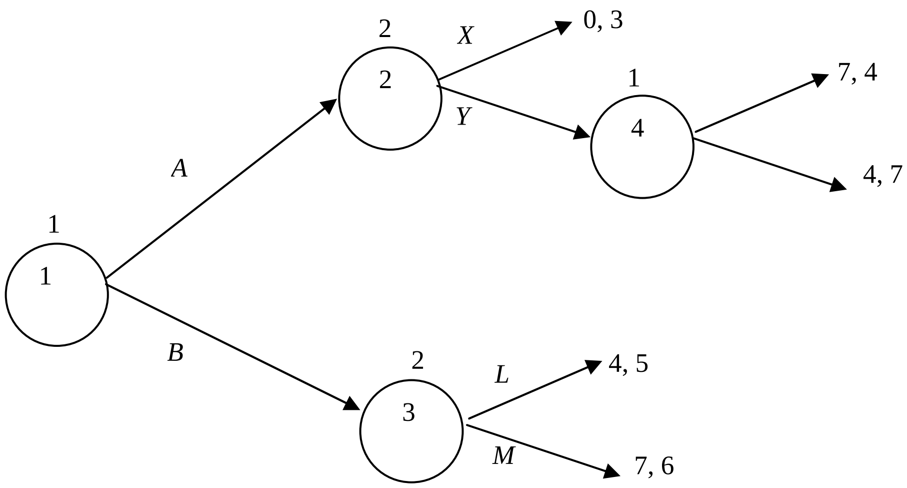

Fig. 7.30

(iv)

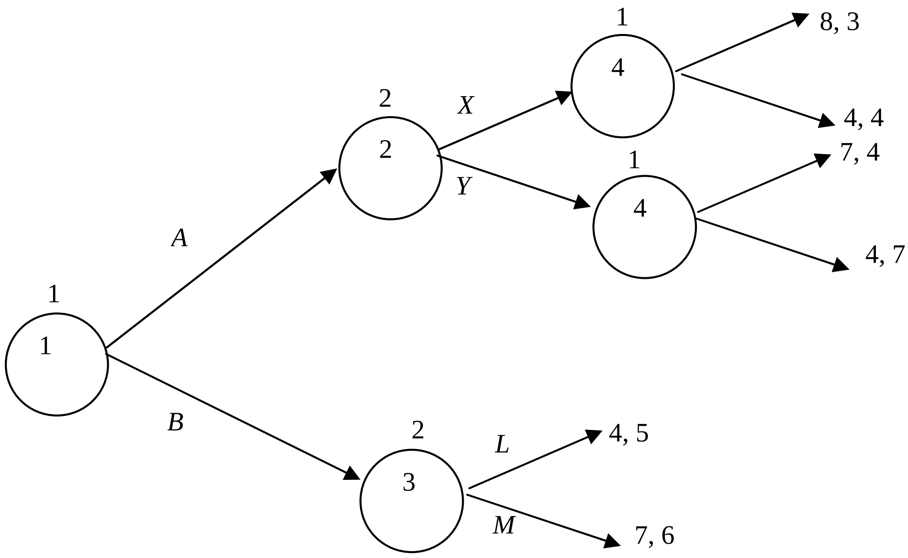

Fig. 7.31

10. For the following game, find if Nash equilibria (if exist) are sub game perfect Nash equilibriua also.

Two players simultaneously and independently choose numbers between 0 and 1. If they both choose 0, they get nothing. If they both choose 1, they get a pay-off of (10, 10). If their choices are different, they again will choose two positive numbers x and y. If $x > y$, their pay-off will be $\left(\frac{x+y}{2}, \frac{x-y}{2}\right)$. If $x < y$, their pay-off will be $\left(\frac{y-x}{2}, \frac{x+y}{2}\right)$. If $x = y$, both get nothing.