Chapter 6

Game Theory- Normal Form Games

Key words: Game theory, strategy, finite game, zero-sum game, pay-off matrix, dominant strategy, value of the game, fair game, stable solution, saddle point, pure strategy, mixed strategy, expected pay-off.

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6.1 Introduction

The maximization of expected value criteria, which we have been discussing till now, is an efficient criterion in the sense that depending upon all the available information, we have to choose a strategy among all possible alternatives, so that we receive the maximum possible benefit. Here we assume that the outcome of a decision is a random variable with some well-defined probability distribution. In other words we say that the outcome of decision is determined by some neutral factor (viz. nature). Obviously, this neutral factor does not have any interest in the benefits or losses, which we are receiving as a participant and hence is not an active participant in the process.

However, the situation may not always be so simple. Sometimes, the outcome of a decision is not controlled by a neutral factor but by a well-informed and intelligent adversary who has an active interest in the process. This is the situation of conflict (and competition).

The world is full of conflicting situations. In fact the resources in the world are limited and when one party tries to increase its share in the available resources, it does so at the cost of any other party. This is the situation of conflict. Have a look at the world surrounding you and you will find conflicts everywhere. Labour-management relationship, political and military conflicts, competitions, maneuvers, marketing and advertising tactics, these all are different faces of conflicts. In all these and many more situations, one party tries to maximize its benefits at the cost of others.

While resolving these disputes is a time-consuming and complex job, it is possible to develop optimal strategies mathematically for such conflicts. Off course, when we are developing strategies, we make some simplifying assumptions, (which we know may not always be true).

6.2 Game Theory

The techniques of developing optimal strategies for dealing with conflicting and competitive situations (whenever these conflicts can be expressed in mathematical terms) have been termed as **game theory**.

We define some terms associated with the game theory

(i) **Strategy** A strategy is a comprehensive plan of action, formulated by a player (an interested and active party in the game), who is well informed of all the alternatives available to him and to his adversary (competing player).

A strategy can be good or bad. The only requirement is that it should be complete and cover all the possibilities.

(ii) **Finite game** When the total number of possible strategies in a game is finite, it is called a finite game. In the other situation, the game is an infinite game.

(iii) **Zero-sum game** Zero-sum games are those games in which one player gains exactly the same amount, which the other player(s) loose so that their net gains is equal to zero.

(iv) **Non zero-sum game** Zero-sum games are those games in which gain of one player is not necessarily equal to the loss of the other or vice versa.

(v) **Pay-off (game) matrix** A pay-off matrix is a tabular representation of the pay-offs of one competitor, which are associated with his strategies in response to the strategies of the other player.

Consider two players A and B playing a zero-sum game. Let A has m strategies numbered $A_1, A_2 \ldots A_m$ available to him and B has n strategies numbered $B_1, B_2 \ldots B_n$, available to him. Let the gain of A, when he chooses i^{th} strategy in response to the j^{th} strategy chosen by B be given by g_{ij}. Then the pay-offs of A can be represented as follows:

$$
G = \begin{array}{c} \\ \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array}
\begin{array}{c} B \\ \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\
\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \vdots & \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{pmatrix} \end{array}
$$

The matrix G is called the pay-off matrix of player A. If $g_{ij} > 0$, A has gained and if $g_{ij} < 0$, then A has lost an amount g_{ij}.

Since the game is a zero-sum game, so whatever is the gain of A is loss of B.

Assumptions of gaming problems Game theory is meant for developing a rational criterion for choosing a strategy among several possible strategies. For developing such criteria, we make some assumptions:

(i) The number of players in the game is (in general) finite.

(ii) The interests of the players clash and each player is choosing his strategy solely for his welfare.

(iii) Each player is well aware of all the strategies available to him and to his opponents.

(iv) All the players are making their moves simultaneously, without knowing the choices, which the other players have made.

(v) The outcome of the game depends upon the moves made by different players; and

(vi) All the players are rational players.

6.3 Solving a zero-sum game

In general, the games are zero-sum games. For the simplicity of presentation, we assume that the games are two- players' games. We define some terms associated with the solution of the games.

(i) Dominant strategy Consider the following game (G_1)

	B		
	B_1	B_2	B_3
A_1	7	4	6
A_2	5	2	4

This game matrix suggests that the two players A and B are playing a game with A having two (viz. A_1 and A_2) and B having three (viz. B_1, B_2 and B_3) strategies. In this case A would always opt for the strategy A_1, as it would yield him better pay-offs than the pay-offs yielded by the strategy A_2. We say that A_1 is a dominant strategy.

A strategy is said to be a dominant strategy if it always yields better (or at least equal) pay-offs than the other strategies irrespective of the strategies opted by the other player(s), i.e., superior strategies (resulting in higher pay-offs) dominate the inferior ones (resulting in lower pay-offs). In such situations, inferior strategies can always be strike off.

Consider the following example

Example 1: Two firms, ABC Ltd. and XYZ Corp. are competitors in the market of electronic goods. In order to increase its market share, each of the firm can opt any of the following three strategies: high advertising, moderate advertising or low advertising. Corresponding to different possible conditions, the pay-offs in terms of percent market share are given below:

		XYZ Corp.		
		High(1)	Moderate(2)	Low(3)
ABC Ltd.	High(1)	2	3	5
	Moderate(2)	2	0	6
	Low(3)	0	2	-1

The managements of the two firms are interested in determining the optimal strategies.

Sol: For XYZ corp., as such there is no dominant strategy. But for ABC Ltd, strategy 1 is dominant over strategy 3. We eliminate the dominated strategy and the reduced pay-off matrix is given by

		XYZ Corp.		
		High(1)	Moderate(2)	Low(3)
ABC Ltd.	High(1)	2	3	5
	Moderate(2)	2	0	6

In this reduced matrix, XYZ corp. would try to minimize its losses so it would eliminate those strategies that are paying a higher pay-off to ABC Ltd. and hence strategies 1 and 2 for XYZ corp. becomes the dominant over strategy 3 and hence strategy 3 can be eliminated. After this elimination we have

		XYZ Corp.	
		High(1)	Moderate(2)
ABC Ltd.	High(1)	2	3
	Moderate(2)	2	0

At this point, ABC Ltd. will again try to maximize its gains and for that it would eliminate strategy 2 so we have

		XYZ Corp.	
		High(1)	Moderate(2)
ABC Ltd.	High(1)	2	3

Finally, XYZ Corp. would settle at strategy 1, which is minimizing the pay-off to ABC Ltd. Hence the optimal strategy for both the players would be to go for high advertising.

With the selection of the optimal strategies, the market share of ABC Ltd would increase by 2%. This is the value of the game.

We define the following terms

Value of the game The pay-off received by the player (whose pay-off matrix is given) when both the players play optimally, is called the value of the game.

Fair game A game that has a value 0, i.e., neither player is neither a loser nor a winner, is called a fair game.

The above game is not a fair game.

In the game G_1, A will always choose the strategy A_1. But what would be the strategy chosen by B? Since whatever is the gain of A, it is the loss of B so naturally B would try to minimize his loss (or gain of A). Then B's obvious choice will be the strategy B_2. The value of the game in this case is 4.

(ii) The maximin and minimax principle Consider the following game:

$$\begin{array}{c|ccc} & & B & \\ & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 3 & 6 \\ A_2 & 2 & 1 & -3 \\ A_3 & 0 & 2 & 1 \end{array}$$

If A chooses A_1, then B will choose B_1 so that the gain of A (or loss of B) is minimized (to 1). For strategy A_2 chosen by A, B will select the strategy B_3. Similarly for strategy A_3 of A, B will choose the strategy B_1. Thus the minimum pay-offs assured to A are

Table 6.1

Strategy	Minimum pay-off
A_1	1
A_2	-3
A_3	0

Among these minimum assured pay-offs A would try to maximize his pay-off. Thus he would choose the strategy A_1 that is the maximin (maximum among minimum) strategy.

Similarly, the player B is playing to minimize his losses. We have the following loss table corresponding to different strategies selected by him:

Table 6.2

Strategy	Minimum pay-off
B_1	2
B_2	3
B_3	6

Then the strategy B_1, which corresponds to the minimum loss among the maximum possible losses, is the minimax strategy.

Thus minimax-maximin principle is that rule on the basis of which one player tries to minimize the worst possible losses and the other player tries to maximize the minimum assured gains.

Example 2: For the firms ABC Ltd. and XYZ Corp., let the pay-off matrix be given by

		XYZ Corp.			
		High(1)	Moderate(2)	Low(3)	Row minimums
ABC Ltd.	High(1)	-4	-1	7	-4
	Moderate(2)	2	0	3	0 ← Maximin value
	Low(3)	3	-2	1	-2
Column maximums		3	0	7	
			↑		
			Minimax value		

On solving this game using dominance principle, we have

Step 1:

		XYZ Corp.	
		High(1)	Moderate(2)
ABC Ltd.	High(1)	-4	-1
	Moderate(2)	2	0
	Low(3)	3	-2

Step 2:

		XYZ Corp.	
		High(1)	Moderate(2)
ABC Ltd.	Moderate(2)	2	0
	Low(3)	3	-2

Step 3:

		XYZ Corp.
		Moderate(2)
ABC Ltd.	Moderate(2)	0
	Low(3)	-2

Step 4:

		XYZ Corp.
		Moderate(2)
ABC Ltd.	Moderate(2)	0

Thus the value of this game is 0. In this game maximin gain of ABC Ltd. is same as the minimax loss of XYZ Corp. Such a game is said to possess a saddle point.

Saddle point A saddle point of a game (if it exists) is that point in the pay-off matrix (of player *A*) where maximin gain of *A* is equal to the minimax loss of player *B*. In such a case, the saddle point is the value of the game.

To find a saddle point

(i) Find the minimum values in the rows;

(ii) Mark the maximum of these minimums. This is the maximin value (or gain) for player *A*;

(iii) Find the maximum values in columns;

(iv) Mark the minimum of these maximums. This is the minimax value (or loss) for player *B*;

(v) If the maximin gain of *A* = the minimax loss of *B*, the point in the pay-off matrix is the saddle point.

Stable (equilibrium) solution If a saddle point exists in a game, the corresponding pair of strategies is a stable (equilibrium) solution.

The solution is stable due to the fact that in presence of a saddle point neither player is in a position to take advantage of the information about the opponent’s choice. In order to optimize his pay-off, each of the players has to stick to that strategy, which would lead him to the saddle point. Hence the solution is a stable solution.

(iii) Use of mixed strategies Consider the following game

		Player B			
		1	2	3	Row minimums
Player A	1	0	-2	2	-2 ← Maximin value
	2	5	4	-3	-3
	3	2	3	-4	-4
Column maximums		5	4	2	
				↑	
				Minimax value	

In this game, since minimax value is not equal to maximin value so the game does not have a saddle point. Since both the players are rational players so they are choosing their strategies in such a manner as to maximize their benefits. Starting with *A*, the player is assured of minimum gain of two units. And subsequently he will choose strategy 3. Anticipating this situation, player *B* will try to minimize the gains of player *A* and hence he will choose that strategy 3. Now player *A*, being aware of the intentions of *B*, will choose strategy 2 to maximize his pay-off. In this position, *B* will have to go for strategy 3 and the whole process will repeat itself. Thus this game does not have a stable solution.

In such games the fact persists that on having information about the opponent's moves; each player can improve his position. Then, in such situations, in order to arrive at a solution, it is necessary that none of the players should have any advance knowledge about the opponent's moves. So, in place of having some criterion about choosing a single strategy to be used definitely, a probability distribution is used to choose among various possible (and acceptable) strategies. By doing so, none of the players will, in advance, be able to know about the strategies to be used by his opponent or to be used by himself.

We define the following terms:

Pure strategy When a player knows with certainty the strategy that he is going to use, the strategy is called a pure strategy. In fact, a pure strategy is a strategy actually used.

Mixed strategy When a player does not know about the strategies to be used with certainty, but uses a probability distribution to determine it, then that probability distribution is called a mixed strategy.

Consider a game, being played by two players A and B that have no saddle point. Let player A has m strategies available to him and the player B has n strategies available to him. Define

$$p_i = P(\text{player } A \text{ uses } i^{th} \text{ strategy}),\ i = \mathbf{1}, \mathbf{2} \ldots m;$$

$$q_j = P(\text{player } B \text{ uses } j^{th} \text{ strategy}),\ j = \mathbf{1}, \mathbf{2} \ldots n.$$

For player A, the pure strategies are $(\mathbf{1}, \mathbf{2} \ldots m)$ and the mixed strategies are $(p_1, p_2 \ldots p_m)$

Expected pay-off: A measure of performance of mixed strategies Let g_{ij} be the pay-off to player A when the player B uses the strategy j and he uses the strategy i, $i = 1,2\ldots m$ and $j = 1,2\ldots n$. Then the expected pay-off of the player A is defined as

$$E(G) = \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}\, p_i q_j$$

Suppose that the mixed strategies used by the player A are $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\right)$ and that used by the player B are $\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$. Then the expected pay-off to player A is,

		Player B		
		1(0)	$2\left(\frac{1}{2}\right)$	$3\left(\frac{1}{2}\right)$
Player A	$1\left(\frac{1}{2}\right)$	0	-2	2
	$2\left(\frac{1}{2}\right)$	5	4	-3
	3(0)	2	3	-4

$$E(G) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-2+2+4-3) = \frac{1}{4}$$

The expected pay-off can be used to choose that strategy which will maximize the minimum guaranteed pay-off (the maximin criterion).

Let the maximin pay-off be denoted by $\underline{v}$.

The optimal strategy for player *B* will minimize the maximum expected loss. Let this value be denoted by $\overline{v}$. In the games, which have no saddle point, if only pure strategies are used, then

$$\underline{v} \leq \overline{v}$$

Thus the players keep on flipping their strategies in a hope to improve their positions.

For a game to be a stable game, it is necessary that

$$\underline{v} = \overline{v}.$$

We state the following theorem

Theorem 6.1 If the mixed strategies are allowed, then the optimal pair of mixed strategies (according to the maximin criterion) will correspond to a stable solution and in that case $\underline{v} = \overline{v} = v$. in this case, v will be the value of the game.

6.4 Calculus approach to solve a 2X2 game

In this method, the expected value of the game for a player is maximized and the probability (mixed strategy) corresponding to which this maximum value exists, is obtained.

Consider the following game

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \left\| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right. \end{array}$$

To find the optimal mixed strategy for this game, we have to optimize expected gain to any one player. Suppose we want to maximize the expected gain to player *A*. We assign probabilities *x* and 1-*x* to *A* of choosing strategy A_1 and A_2 respectively. Similarly probabilities *y* and 1-*y* are assigned to player *B* for choosing strategies B_1 and B_2 respectively. Then expected gain to player *A* is given by

$$E(A) = xya_{11} + x(1-y)a_{12} + (1-x)ya_{21} + (1-x)(1-y)a_{22}$$

To determine optimal values of x and y, we differentiate this function partially with respect to x and y at 0,0, and we get

$$\frac{\partial E(A)}{\partial x} = ya_{11} + (1-y)a_{12} - ya_{21} - (1-y)a_{22} = 0$$

and

$$\frac{\partial E(A)}{\partial y} = xa_{11} + (1-x)a_{21} - xa_{12} - (1-x)a_{22} = 0$$

The two equations yield

$$y(a_{11} + a_{22} - a_{12} - a_{21}) = a_{22} - a_{12}$$

$$\Rightarrow y = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

and

$$x(a_{11} + a_{22} - a_{12} - a_{21}) = a_{22} - a_{21}$$

$$\Rightarrow x = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

Substituting these values of x and y in E (A), we have

$$E(A) = xya_{11} + x(1-y)a_{12} + (1-x)ya_{21} + (1-x)(1-y)a_{22}$$

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

For second order conditions, we have

$$\frac{\partial^2 E(A)}{\partial x^2} = 0; \quad \frac{\partial^2 E(A)}{\partial y^2} = 0; \quad \text{and}$$

$$\frac{\partial^2 E(A)}{\partial y \partial x} = a_{11} + a_{22} - a_{21} - a_{12}$$

$$\Rightarrow \left(\frac{\partial^2 E(A)}{\partial y \partial x}\right)^2 - \left(\frac{\partial^2 E(A)}{\partial x^2}\right)\left(\frac{\partial^2 E(A)}{\partial y^2}\right) > 0$$

Thus expected gain of player A for these values of x and y is maximum.

We illustrate the results with the help of the following examples.

Example 3: Let the pay-off matrix of two players A and B is given as follows:

$$\begin{array}{ccc|cc} & & & \multicolumn{2}{c}{\text{Player } B} \\ & & & y & 1\text{-}y \\ \hline \text{Player } A & x & & 1 & 7 \\ & 1\text{-}x & & 5 & 2 \end{array}$$

Player A selects his strategies with probabilities x and 1-x respectively and the player B chooses his strategies with probabilities y and 1-y respectively. Find the solution to the game.

Sol: Expected gain of player A is

$$E(A) = xy + 7x(1-y) + 6(1-x)y + 2(1-x)(1-y)$$

This expected gain is to be maximized for player A. for that, we differentiate the expression partially with respect to x and y at zero. We have

$$\frac{\partial E(A)}{\partial x} = y + 7(1-y) - 6y - 2(1-y) = 0$$

$$\Rightarrow -10y + 5 = 0$$

$$\Rightarrow y = 1 - y = \frac{1}{2}$$

And

$$\frac{\partial E(A)}{\partial y} = x - 7x + 6(1-x) - 2(1-x) = 0$$

$$\Rightarrow -10x + 4 = 0$$

$$\Rightarrow x = \frac{2}{5} \text{ and } 1 - x = \frac{3}{5}$$

The value of the game is

$$\begin{aligned} E(A) &= xy + 7x(1-y) + 6(1-x)y + 2(1-x)(1-y) \\ &= 4.0 \end{aligned}$$

Example 4: In a coin-tossing game, Player A wins a unit amount when there is a sequence of HH and wins nothing if there is a sequence of TT. However, a sequence of HT or TH results in a loss of half unit to player A. Find the best strategies for two players and the value of the game.

Sol: The pay-off matrix of the player *A* is as given below:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{Player } B} \\ & & B_1 & B_2 \\ \hline \text{Player } A & A_1 & 1 & -0.5 \\ & A_2 & -0.5 & 0 \end{array}$$

Since the game does not possess any saddle point, we use mixed strategies for two players. Let Player *A* selects his strategies with probabilities *x* and 1-*x* respectively and the player *B* chooses his strategies with probabilities *y* and 1-*y* respectively. Then

$$x = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{0.5}{1 + 0.5 + 0.5} = 0.25$$

$$\text{and } y = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{0.5}{1 + 0.5 + 0.5} = 0.25$$

$$\text{Then } 1\text{-}x = 1\text{-}y = 0.75$$

The expected value of the game is

$$\upsilon = \frac{-(0.5)(0.5)}{1 + 0.5 + 0.5} = -\frac{0.25}{2} = -0.125$$

6.5 Graphical solution procedure

Consider again the above pay-off matrix:

$$\begin{array}{cc|ccc} & & \multicolumn{3}{c}{\text{Player } B} \\ & & 1 & 2 & 3 \\ \hline & 1 & 0 & -2 & 2 \\ \text{Player } A & 2 & 5 & 4 & -3 \\ & 3 & 2 & 3 & -4 \end{array}$$

Foe player *A*, strategy 2 is dominant over strategy 3, so after eliminating strategy 3, the mixed strategies for player *A* are (p_1, p_2); $p_2 = \mathbf{1} - p_1$ and the reduced pay-off matrix is

		Player B		
		$1(q_1)$	$2(q_2)$	$3(q_3)$
	$1(p_1)$	0	-2	2
Player A	$2(1-p_1)$	5	4	-3

If player B chooses strategy 1, then the expected pay-off of player B is $\mathbf{0}p_1 + (\mathbf{1} - p_1)\mathbf{5} = \mathbf{5} - \mathbf{5}p_1$. Similarly for strategy 2, the expected pay-off is $-\mathbf{2}p_1 + (\mathbf{1} - p_1)\mathbf{4} = \mathbf{4} - \mathbf{6}p_1$ and that for strategy 3 is $\mathbf{2}p_1 - (\mathbf{1} - p_1)\mathbf{3} = -\mathbf{3} + \mathbf{5}p_1$.

In graphical method of solving a game problem, we plot these expected pay-offs on a graph.

Player B is playing to minimize the expected pay-off of player A, so he will choose that strategy, which corresponds to the lines lying in the bottom part of the graph, i.e., $-\mathbf{3} + \mathbf{5}p_1$ or $\mathbf{4} - \mathbf{6}p_1$.

Now, the player A is playing to maximize this expected pay-off so he chooses that strategy which will maximize the pay-off yielded from these lines, i.e. the intersection of these two lines which is given by

$$-\mathbf{3} + \mathbf{5}p_1 = \mathbf{4} - \mathbf{6}p_1$$

$$\Rightarrow p_1 = \frac{\mathbf{7}}{\mathbf{11}} \text{ and } p_2 = \mathbf{4}$$

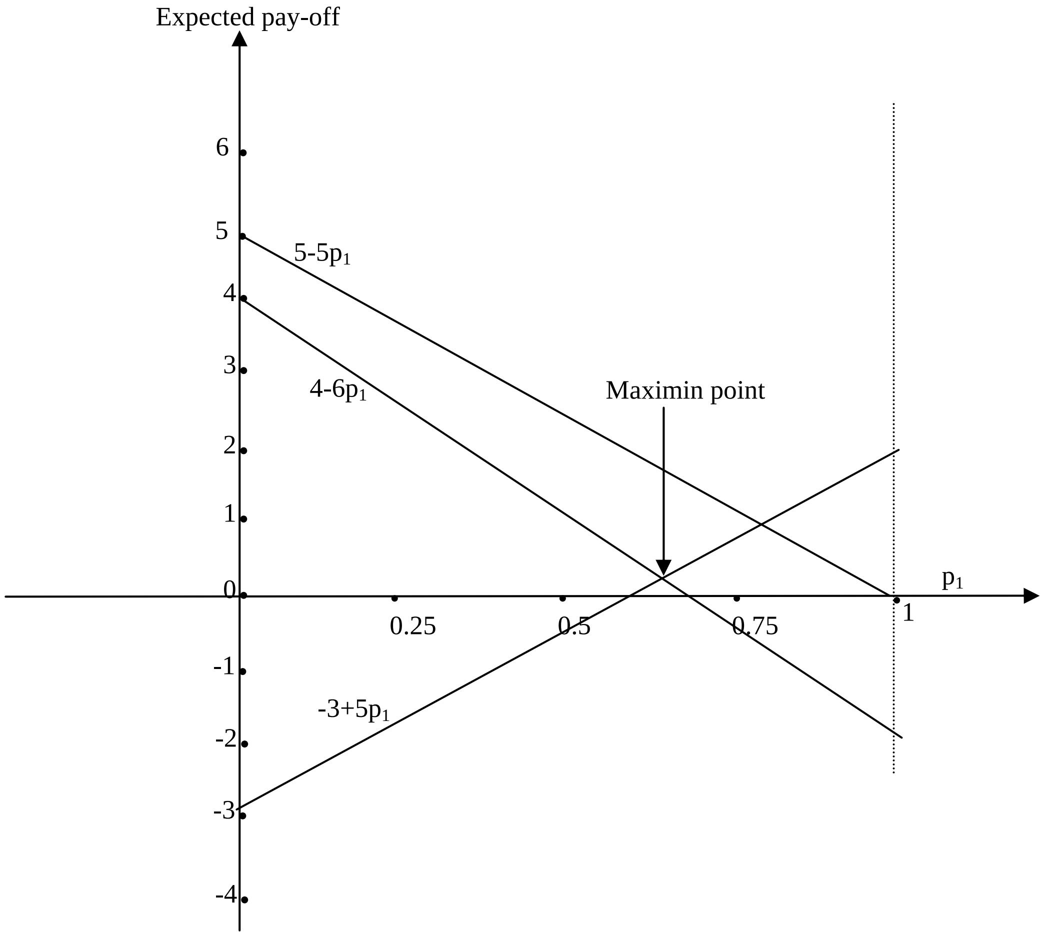

Fig. 6.1

Thus, the optimal strategies for player *A* are $\left(\frac{\mathbf{7}}{\mathbf{11}}, \frac{\mathbf{4}}{\mathbf{11}}, \mathbf{0}\right)$ and the value of the game is

$$-\mathbf{3} + \mathbf{5}\left(\frac{\mathbf{7}}{\mathbf{11}}\right) = \frac{\mathbf{2}}{\mathbf{11}}$$

Now, we proceed to find the optimal strategies for player *B*:

The expected pay-off of player *A* is $q_1(\mathbf{5} - \mathbf{5}p_1) + q_2(\mathbf{4} - \mathbf{6}p_1) + q_3(-\mathbf{3} + \mathbf{5}p_1)$ and by the minimax theorem as $\underline{\nu} = \bar{\nu} = \nu$, so if the optimal strategies for the player *B* are (q'_1, q'_2, q'_3), these must satisfy the maximum expected pay-off to A, I.e.,

$$q'_1(\mathbf{5} - \mathbf{5}p_1) + q'_2(\mathbf{4} - \mathbf{6}p_1) + q'_3(-\mathbf{3} + \mathbf{5}p_1) \leq \bar{\nu} = \nu = \frac{\mathbf{2}}{\mathbf{11}}$$

$$\mathbf{0} \leq p_1 \leq 1$$

Put $p_1 = \frac{7}{11}$ so

$$\frac{20}{11}q'_1 + \frac{2}{11}q'_2 + \frac{2}{11}q'_3 = \frac{2}{11}$$

$$\text{Also} \quad q'_1 + q'_2 + q'_3 = 1$$

Since $\frac{20}{11} > \frac{2}{11}$ and $q'_1 \geq 0$, so the only possible value for q'_1 is $q'_1 = 0$,

So we have

$$q'_2(4-6p_1) + q'_3(-3+5p_1) \begin{cases} \leq \frac{2}{11}; 0 \leq p_1 \leq 1 \\ = \frac{2}{11}; \; p_1 = \frac{7}{11} \end{cases}$$

As the left hand side is a linear expression so the only possible solution to this relation is

$$q'_2(4-6p_1) + q'_3(-3+5p_1) = \frac{2}{11}$$

Let $p_1 = 0$ and 1,

$$\Rightarrow \qquad 4q'_2 - 3q'_3 = \frac{2}{11}$$

$$\text{and} \qquad -2q'_2 + 2q'_3 = \frac{2}{11}$$

Solving these two equations, we have

$$q'_2 = \frac{5}{11} \quad \text{and} \quad q'_3 = \frac{6}{11}$$

Thus the optimal strategies for player *B* are $\left(0, \frac{5}{11}, \frac{6}{11}\right)$.

6.6 Linear programming approach

The graphical method, although a convenient method of solving a game problem, which does not have a saddle point, suffers from a major limitation. This method can be used only when one of the players has only two (undominated) pure strategies. But, this may be too simplified a situation. In large games, where each of the players may have several strategies to choose from, this method cannot be employed conveniently.

In such problems, linear programming approach is adopted.

The expected pay-off to player A is

$$E(G) = \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij} p_i q_j$$

For a strategy $(p_1, p_2, ..., p_m)$ to be optimal, $E(G) \; = \; \underline{v} \; = \; v$.

Since this relationship holds for each of the opponent's strategy $(q_1, q_2, ..., q_n)$ so we have,

$$\text{for } j = 1, 2 ... n, \text{ and putting } q_j = 1 \;\&\; q_k = 0 \;\forall\; k \neq j$$

$$\sum_{i=1}^{m} g_{ij} p_i \;\geq\; v$$

Also,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij} p_i q_j \;=\; \sum_{j=1}^{n} q_j \sum_{i=1}^{m} g_{ij} p_i$$

$$\geq \; \sum_{j=1}^{n} q_j v \;=\; v \;\text{ as } \sum_{j=1}^{n} q_j \;=\; 1$$

Then the problem can be restated as follows

$$\text{maximise } p_{m+1} \;\; (=v)$$

subject to

$$g_{11} p_1 + g_{21} p_2 + ... + g_{m1} p_m - p_{m+1} \;\geq\; 0$$

$$g_{12} p_1 + g_{22} p_2 + ... + g_{m2} p_m - p_{m+1} \;\geq\; 0$$

$$\vdots$$

$$g_{1n} p_1 + g_{2n} p_2 + ... + g_{mn} p_m - p_{m+1} \;\geq\; 0$$

$$p_1 + p_2 + ... + p_m \;=\; 1$$

$$p_i \;\geq\; 0 \;\;\forall\;\; i = 1, 2 ... m.$$

For player B, the equivalent problem is

$$\text{minimise } q_{m+1} \ (\ = v)$$

subject to

$$g_{11}q_1 + g_{12}q_2 + \ldots + g_{1n}q_n - q_{n+1} \quad \le \quad 0$$

$$g_{21}q_1 + g_{22}q_2 + \ldots + g_{2n}q_m - q_{n+1} \quad \le \quad 0$$

$$\vdots$$

$$g_{m1}q_1 + g_{m2}q_2 + \ldots + g_{mn}q_m - q_{n+1} \quad \le \quad 0$$

$$q_1 + q_2 + \ldots + q_n \quad = \quad 1$$

$$q_i \quad \ge \quad 0 \ \ \forall \ \ i = 1, 2 \ldots n$$

Thus the two problems are dual problems.

In order to ensure that p_{m+1} and q_{n+1} are non-negative, a sufficiently large quantity can be added to it. As this quantity is added to every entry, the optimal strategy will remain the same.

To demonstrate this approach, we again consider the earlier problem, which using linear programming approach can be restated as follows:

$$\text{maximise } p_3 \ (\ = v)$$

subject to

$$5p_2 - p_3 \quad \ge \quad 0$$

$$-2p_1 + 4p_2 - p_3 \quad \ge \quad 0$$

$$2p_1 - 3p_2 - p_3 \quad \ge \quad 0$$

$$p_1 + p_2 \quad = \quad 1$$

$$p_i \quad \ge \quad 0 \quad i = 1, 2.$$

The following algorithm gives the optimal solution to the problem

Simplex method- Two-phase procedure

Phase 1- Iteration 1 **Table 6. 3**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	1.00	7.00	3.00	-1.00	-1.00	-1.00	0.00	0.00	0.00	0.00	1.00
Rp_7	0.00	5.00	1.00	-1.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
Rp_8	-2.00	4.00	1.00	0.00	-1.00	0.00	0.00	1.00	0.00	0.00	0.00
Rp_9	2.00	-3.00	1.00	0.00	0.00	-1.00	0.00	0.00	1.00	0.00	0.00
Rp_{10}	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y/n*)?	*n*	*n*	*y*								

Phase 1- Iteration II **Table 6. 4**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	1.00	0.00	1.60	0.40	-1.00	-1.00	-1.40	0.00	0.00	0.00	1.00
p_2	0.00	1.00	0.20	-0.20	0.00	0.00	0.20	0.00	0.00	0.00	0.00
Rp_8	-2.00	4.00	0.20	0.80	-1.00	0.00	-0.80	1.00	0.00	0.00	0.00
Rp_9	2.00	0.00	1.60	-0.60	0.00	-1.00	0.60	0.00	1.00	0.00	0.00
Rp_{10}	1.00	0.00	-0.20	0.20	0.00	0.00	-0.20	0.00	0.00	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y/n*)?	*n*	*n*	*y*								

Phase 1- Iteration III **Table 6. 5**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	1.00	-8.00	0.00	2.00	-1.00	-1.00	-3.00	0.00	0.00	0.00	1.00
p_3	0.00	5.00	1.00	-1.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
Rp_8	-2.00	-1.00	0.00	1.00	-1.00	0.00	-1.00	1.00	0.00	0.00	0.00

Rp_9	2.00	-8.00	0.00	1.00	0.00	-1.00	-1.00	0.00	1.00	0.00	0.00
Rp_{10}	1.00	1.00	0.00	0.00	0.00	0.00	-0.20	0.00	0.00	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y*/*n*)?	*n*	*n*	*y*								

Phase 1- Iteration IV **Table 6. 6**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	5.00	-6.00	0.00	0.00	1.00	-1.00	-1.00	-2.00	0.00	0.00	1.00
p_3	-2.00	4.00	1.00	0.00	-1.00	0.00	1.00	1.00	0.00	0.00	0.00
Sp_4	-2.00	-1.00	0.00	1.00	-1.00	0.00	-1.00	1.00	0.00	0.00	0.00
Rp_9	4.00	-7.00	0.00	0.00	1.00	-1.00	0.00	-1.00	1.00	0.00	0.00
Rp_{10}	1.00	1.00	0.00	0.00	0.00	0.00	0.20	0.00	0.00	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y*/*n*)?	*n*	*n*	*y*								

Phase 1- Iteration V **Table 6. 7**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	0.00	2.75	0.00	0.00	-0.25	0.25	-1.00	-0.75	-1.25	0.00	1.00
p_3	0.00	0.50	1.00	0.00	-0.50	-0.50	0.00	0.50	0.50	0.00	0.00
Sp_4	0.00	-4.50	0.00	1.00	-0.50	-0.50	-1.00	0.50	0.50	0.00	0.00
p_1	1.00	-1.75	0.00	0.00	0.25	-0.25	0.00	-0.25	0.25	0.00	0.00
Rp_{10}	0.00	2.75	0.00	0.00	-0.25	0.25	0.00	0.25	-0.25	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y*/*n*)?	*n*	*n*	*y*								

Phase 1- Iteration VI **Table 6. 8**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	0.00	0.00	5.50	0.00	2.50	3.00	-1.00	-3.50	-4.00	0.00	1.00
p_2	0.00	1.00	-2.00	0.00	-1.00	-1.00	0.00	1.00	1.00	0.00	0.00
Sp_4	0.00	0.00	-9.00	1.00	-5.00	-5.00	-1.00	5.00	5.00	0.00	0.00
p_1	1.00	0.00	-3.50	0.00	-1.50	-2.00	0.00	1.50	2.00	0.00	0.00
Rp_{10}	0.00	0.00	5.50	0.00	2.50	3.00	0.00	-2.50	-3.00	1.00	1.00
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (y/n)?	n	n	y								

Phase 1- Iteration VII **Table 6. 9**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Solution
z (min)	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	1.00	-1.00	-1.00	0.00
p_2	0.00	1.00	0.00	0.00	-0.09	0.09	0.00	0.09	-0. 09	0.36	0.36
Sp_4	0.00	0.00	0.00	1.00	-0.91	-0.09	-1.00	0.91	0. 09	1.64	1.64
p_1	1.00	0.00	0.00	0.00	0.09	-0.09	0.00	-0.09	0.09	0.64	0.64
p_3	0.00	0.00	1.00	0.00	0.45	0.55	0.00	-0.45	-0.55	0.18	0.18
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (y/n)?	n	n	y								

Phase 2- Iteration I **Table 6. 10**

Basic variables	p_1	p_2	p_3	Sp_4	Sp_5	Sp_6	Rp_7	Rp_8	Rp_9	Rp_{10}	Sol
z (min)	0.00	0.00	0.00	0.00	0.45	0.55	blocked	blocked	blocked	blocked	0.18
p_2	0.00	1.00	0.00	0.00	-0.09	0.09	0.00	0.09	-0. 09	0.36	0.36
Sp_4	0.00	0.00	0.00	1.00	-0.91	-0.09	-1.00	0.91	0. 09	1.64	1.64
p_1	1.00	0.00	0.00	0.00	0.09	-0.09	0.00	-0.09	0.09	0.64	0.64

p_3	0.00	0.00	1.00	0.00	0.45	0.55	0.00	-0.45	-0.55	0.18	0.18
Lower bound	0.00	0.00	$-\infty$								
Upper bound	∞	∞	∞								
Unrestricted (*y/n*)?	*n*	*n*	*y*								

Thus the optimal mixed strategies are (0.64, 0.36, 0) and the value of the game is 0.18.

The corresponding dual problem is

$$\text{minimise } q_4 \;\; (\;=v)$$

subject to

$$-2q_2 + .2q_3 - q_4 \quad \leq \quad 0$$

$$5q_1 + 4q_2 + -3q_3 - q_4 \quad \leq \quad 0$$

$$q_1 + q_2 + q_3 \quad = \quad 1$$

$$q_i \quad \geq \quad 0 \quad i = 1, 2, 3$$

This can be solved using duality criterion. However, an independent solution has been provided below

Phase 1- Iteration I **Table 6. 11**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	1.00	1.00	1.00	0.00	0.00	0.00	0.00	1.00
Sq_5	0.00	-2.00	2.00	-1.00	1.00	0.00	0.00	0.00
Sq_6	5.00	4.00	-3.00	-1.00	0.00	1.00	0.00	0.00
Rq_7	1.00	1.00	1.00	0.00	000	0.00	1.00	1.00
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (*y/n*)?	*n*	*n*	*n*	*y*				

Phase 1- Iteration II **Table 6. 12**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	0.00	0.2.	1.60	0.20	0.00	-0.20	0.00	1.00
Sq_5	0.00	-2.00	2.00	-1.00	1.00	0.00	0.00	0.00
q_1	1.00	0.80	-0.60	-0.20	0.00	0.20	0.00	0.00
Rq_7	1.00	0.20	1.60	0.20	000	-0.20	1.00	1.00
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (y/n)?	n	n	n	y				

Phase 1- Iteration III **Table 6. 13**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	0.00	1.80	0.00	1.00	-0.80	-0.20	0.00	1.00
q_3	0.00	-1.00	1.00	-0.50	0.50	0.00	0.00	0.00
q_1	1.00	0.20	0.00	-0.50	0.30	0.20	0.00	0.00
Rq_7	0.00	1.80	0.00	1.00	-0.80	-0.20	1.00	1.00
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (y/n)?	n	n	n	y				

Phase 1- Iteration IV **Table 6. 14**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	-9.00	0.00	0.00	5.50	-0.80	-2.00	0.00	1.00
q_3	5.00	0.00	1.00	-3.00	0.50	1.00	0.00	0.00
q_2	5.00	1.00	0.00	-2.50	0.30	1.00	0.00	0.00
Rq_7	-9.00	0.00	0.00	5.50	-0.80	-2.00	1.00	1.00
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (y/n)?	n	n	n	y				

Phase 1- Iteration V **Table 6. 15**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	0.00	0.00	0.00	0.00	-0.80	0.00	-1.00	0.00
q_3	0.09	0.00	1.00	0.00	0.50	-0.09	0.55	0.55
q_2	0.91	1.00	0.00	0.00	0.30	0.09	0.45	0.45
q_4	-1.64	0.00	0.00	1.00	-0.80	-0.36	0.18	0.18
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (y/n)?	n	n	n	y				

Phase 2- Iteration 1 **Table 6. 16**

Basic variables	q_1	q_2	q_3	q_4	Sq_5	Sq_6	Rq_7	Solution
z (min)	-1.64	0.00	0.00	0.00	-0.64	-0.36	blocked	0.18
q_3	0.09	0.00	1.00	0.00	0.09	-0.09	0.55	0.55
q_2	0.91	1.00	0.00	0.00	-0.09	0.09	0.45	0.45
q_4	-1.64	0.00	0.00	1.00	-0.64	-0.36	0.18	0.18
Lower bound	0.00	0.00	0.00	$-\infty$				
Upper bound	∞	∞	∞	∞				
Unrestricted (y/n)?	n	n	n	y				

Thus the optimal mixed strategies are (0, 0.45, 0.55) and the value of the game is 0.18.

6.7 Non Zero-sum games

As we have seen above, in general, in competitive games, the two players' pay-offs sum to zero. For example, consider the following famous games of matching pennies.

Two players have two coins covered by their hands. Simultaneously and independently, they select a "head" or a "tail" by uncovering the coins in their hands. If the selections match, player 2 will give his coin to player 1 and if the selections mismatch, player 1 will give his coin to player 2. Then the pay-off matrix of the game would be

		Player 2	
		H	*T*
Player 1	*H*	(1,-1)	(-1,1)
	T	(-1,1)	(1,-1)

In any case, the sum of the pay-offs of the two players is zero.

However consider the following game

		Player 2	
		1	2
Player 1	1	(4,3)	(1,4)
	2	(6,2)	(2,3)

In this case, the players' pay-offs do not add up to zero. Such a game is called a **non zero-sum** game.

In order to solve this game, we notice that whatever strategy player A chooses, his pay-off will be less than that of B. So what should he do? He should choose a strategy which will maximize his minimum pay-off. So he should choose strategy 1. If the choice lies with B, he will go for strategy 2.

Problems

1. Consider a game with the following pay-off matrix

Player *B*

		1	2	3	4	5
	1	3	-1	4	6	7
Player *A*	2	-1	8	2	4	12
	3	16	8	6	14	12
	4	1	11	-4	2	1

Find the saddle point of the game and also find the value of the game.

2. Find the game whose pay-off matrix is given below

Player *B*

		1	2	3
	1	2	3	0.5
Player *A*	2	1.5	2	0
	3	0.5	1	1

3. Using minimax criterion, find the solution to the following game

Player *B*

		1	2	3	4
	1	3	-3	-2	-4
Player *A*	2	-4	-2	-1	1
	3	1	=	2	0

4. Two players A and B are playing a game with five rupees, ten rupees, and a twenty rupees note. Each player selects a note without the knowledge of the other. If the sum of the notes selected is even, A gets B's note and if the sum is odd B gets A's note. Find the best strategy for each player and the value of the game.

5. Using graphical procedure, find the value of the following game

		Player *B*			
		1	2	3	4
Player *A*	1	1	0	-1	-1
	2	-3	-2	1	-2

6. Two companies A and B are competing for the same product in the market. The pay-offs corresponding to different strategies are given in the following pay-off matrix. Find the solution to the game by the linear programming approach.

		Company *B*		
		1	2	3
Company *A*	1	2	-2	3
	2	-3	5	-1

7. Solve the game whose pay-off matrix is given below

		Player *B*		
		1	2	3
Player *A*	1	1	3	1
	2	0	-4	-3
	3	1	5	-1

8. Determine the range of p and q that will make the pay-off element a_{22} a saddle point for the game whose pay-off matrix is given below

		Player *B*		
		1	2	3
Player *A*	1	2	4	7
	2	10	7	q
	3	4	p	8

9. Consider a "modified" form of "matching biased coins" game problem being played by two players. If a run of size two occurs, player *A* is a winner otherwise player *B* is winner. In this game, player *A* gets Rs. 8.00 if a run of two heads appears; and he gets Rs. 1.00 if a run of two tails appears. Player *B* is paid Rs. 3.00 if a mismatch occurs. Given the choice of being

matching or mismatching player, which alternative should one opt for and what should be the optimal strategy?

9. Obtain the optimal strategies for both persons and the value of the game for the game with the pay-off matrix

		Player *B* 1	2
Player *A*	1	1	-3
	2	3	5
	3	-1	6
	4	4	1
	5	2	2
	6	-5	6

10. Two firms are competing for business under the condition so that one firm's gain is the other's loss. Firm *A*'s pay-off matrix is given below

		Firm *B* No	Medium	Heavy
Firm *A*	No advertising	10	5	-2
	Medium advertising	13	12	15
	Heavy advertising	16	14	10

What should be the optimal strategies and what is the value of the game?