Chapter 2

Decision making under risk: No data problem

Key words: Decisions, pay-off, prior distribution, likelihood, posterior distribution, normal distribution, beta distribution, linear pay-off function, two action problem, expected value of perfect information, decision making and point estimation

Suggested readings:

1. Winkler R. L. and Hays W. L. (1975), **Statistics: Probability, Inference, and Decision** (2nd edition), Holt, Rinehart and Winston
2. Frame P. and Taylor R. (1776), **Statistical Analysis for Business Decisions**, McGraw-Hill
3. Berger J. O. (1980), **Statistical decision Theory: Foundations, Concepts and Methods**, Springer- Verlag.
4. Goon A.M., Gupta M.K. and Dasgupta B, (1987), **An Outline of Statistical Theory (volume II)**, The World Press Private Ltd.

2.1 Introduction

We have seen that making decisions under risk is a situation where there exists a set of states of nature having non-zero probabilities associated with the process.

In fact, statistical inference is the process of making decisions in the face of uncertainty. For example, in point estimation, the problem is that of deciding a value in the parameter space as the true value of the parameter. Similarly, in testing of hypothesis, a decision is taken whether to accept or reject a hypothesis in the presence of given sample.

The inferential procedures of decision making are based on sampling information. In general, some assumptions are made regarding the population or the process under investigation. These assumptions are needed to determine sampling distributions of the sampling statistics under consideration, such as sample mean or sample variance. This approach of decision making is called ***classical statistical inference*** or ***sampling theory approach***. The approach is empirical in nature as it just depends on the available sample information.

In classical inference, a decision making problem can be formulated as follows:

Let a state of nature be denoted by θ and Θ denotes the state space of all the possible states of nature. Similarly, let d denotes particular decision from the decision space D. Further, denote by L (θ, d) the loss function of the loss incurred as a result of taking decision d when the true state of nature is θ. We have assumed that the loss can be expressed numerically.

Let $X_1, X_2, ..., X_n$ be the set of random variables defined on the real line $\Re$ and a realized set of observations on theses random variables be denoted by $x_1, x_2, ..., x_n$. This is on the basis of this set; a decision is to be taken.

A decision procedure (or a decision function or a decision rule) is a mapping from $\Re^n$ to D, i.e. if we denote the decision function by $\delta(\underset{\sim}{x}) = \delta(x_1, x_2, ..., x_n)$ then

$$\delta(\underset{\sim}{x}): \ \Re^n \rightarrow \mathrm{D}$$

such that $\delta(\underset{\sim}{x})$ is optimal in a class of decision functions, say, Δ.

The performance of a decision function can be expressed in terms of the loss function L $(\theta, \underset{\sim}{x})$ or in terms of decision rule $\delta(\underset{\sim}{x})$, by L $(\theta, \delta(\underset{\sim}{x}))$. The expected value of the loss function is the ***risk*** associated with the decision rule. Mathematically, risk associated with the decision rule $\delta(\underset{\sim}{x}) \in \Delta$ can be defined as

$$r(\theta, \delta(\underset{\sim}{x})) = E_\theta(\mathrm{L}(\theta, \delta(\underset{\sim}{x}))), \quad \theta \in \Theta$$

A decision rule $\delta(.) \in \Delta$ is said to be **optimal** (or uniformly best or the most desirable) decision rule if

$$r(\theta, \delta) \leq r(\theta, \delta') \, \forall \theta \in \Theta \text{ and any } \delta' \in \Delta$$

A decision rule $\delta(.) \in \Delta$ is said to be an **admissible** decision rule if

$$r(\theta, \delta) \leq r(\theta, \delta') \, \forall \theta \in \Theta; \quad \text{and}$$
$$r(\theta, \delta) < r(\theta, \delta') \text{ for some } \theta \in \Theta \text{ for any } \delta' \in \Delta$$

In other words, a decision rule $\delta(.) \in \Delta$ is an admissible decision rule if $\nexists$ a $\delta' \in \Delta$ with a smaller uniform risk.

2.2 Bayesian approach of decision making

In classical inference, decisions are made on the basis of available sample information only and any other information, if there is any, is only utilized informally, e.g. in deciding the use of a particular model such as a Bernoulli or a normal model.

There is another approach to decision making which utilizes the other available information in a formal manner. This approach, based on Bayesian theorem, is called the Bayesian Inference.
Now, suppose that θ (the state of nature) is the parameter about which we wish to make inference. There is some information available about the parameter θ which can be expressed in terms of the probability distribution

$$P(\theta = \theta_i) \quad \forall \quad \theta_i \in \Theta$$

This probability distribution is called the **prior probability distribution** of the parameter θ.

Let $\boldsymbol{Y}$ be the sampling statistic under consideration. Denote by $P(\boldsymbol{y} \mid \theta)$ the likelihood function of the observed value $\boldsymbol{y}$ of $\boldsymbol{Y}$ given θ.

Then the Bayes' theorem says that **the conditional probability distribution of θ given y (*the posterior distribution of θ*) is given by**

$$P(\theta_i \mid y) \;=\; \frac{P(y\mid\theta_i)P(\theta_i)}{\sum_{j=1}^{n} P(y\mid\theta_i)P(\theta_i)}, \quad \theta_i \in \Theta \;\forall i = 1,2,\ldots n \text{ if } \theta \text{ is discrete}$$

$$= \frac{P(y\mid\theta_i)P(\theta_i)}{\int_{\theta} P(y\mid\theta)P(\theta)d\theta} \text{ ,if } \theta \text{ is continuous}$$

Thus in order to compute the distribution of the parameter θ in the light of given sample $\boldsymbol{y}$, we make use of the previous information about θ also which may be non-sample (non - objective) information. Putting in other words, Bayesian approach of inference is based on making use of all available information, be it sampling or non-sampling in nature.

This characteristic of Bayesian inference makes it a strong tool in decision theory as the decisions are costly and sometimes irreversible in nature and as such it may not be advisable to ignore any information, in spite of the from in which it is available.

We now define risk in terms of Bayesian terminology.

Risk: Let $f_\theta(.)$ denotes the p.d.f. (p.m.f.) of $X_1, X_2, \ldots, X_n$ for parameter θ. Then

$$r\,((\theta, \delta) \;=\; \int_\Theta \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x)\,dx \quad \text{if } \boldsymbol{X} \text{ is continuous; and}$$

$$= \sum \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x) \quad \text{if } \boldsymbol{X} \text{ is discrete.}$$

If $\xi(.)$ is the p.d.f. (p.m.f.) of θ, then the average risk is

$$r\,(\xi, \delta) \;=\; \int_\theta \int_X \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x)\,\xi(\theta)\,dx\,d\theta; \text{ if } \boldsymbol{X} \text{ is continuous and } \theta \text{ is continuous;}$$

$$= \int_\theta \sum_X \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x)\,\xi(\theta)\,d\theta \quad \text{if } \boldsymbol{X} \text{ is discrete and } \theta \text{ is continuous.}$$

$$= \sum_\Theta \int_X \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x)\,\xi(\theta)\,dx \quad \text{if } \boldsymbol{X} \text{ is continuous and } \theta \text{ is discrete.}$$

$$= \sum_\Theta \sum_X \mathrm{L}\,(\theta, \delta(\underset{\sim}{x}))\; f_\theta(x)\,\xi(\theta) \quad \text{if } \boldsymbol{X} \text{ is dicsrete and } \theta \text{ is discrete}$$

Risk may also be defined in terms of the posterior distribution of θ as follows:

Let $$f_\xi(x) = \int_\theta f_\theta(x)\,\xi(\theta)\,d\theta$$

Then $\zeta_x(\theta) = \dfrac{f_\theta(x)\xi(\theta)}{f_\xi(x)}$ $\forall$ x such that $f_\xi(x) > 0$ is the posterior distribution of θ corresponding to the prior distribution $\xi(.)$. Then

$$r(\xi,\delta) = \int_\theta \int_X \mathrm{L}(\theta, \delta(\underset{\sim}{x}))\ \zeta_x(\theta)\ d\theta; \qquad \text{if } \boldsymbol{X} \text{ is continuous and } \theta \text{ is continuous;}$$

The risk may similarly be defines if X is discrete and θ is discrete or continuous

2.3 Point estimation as a decision problem

Consider a function $\tau(\theta)$ of θ, $\theta \in \Theta$ to be estimated on the basis of observations on the set of random variables $X_1, X_2, ..., X_n$.

A decision rule $\xi \in$ D on $X_1, X_2, ..., X_n$ or $\boldsymbol{X}$ is said to be an estimator of $\tau(\theta)$ if for any set of observations $\boldsymbol{x}$ on $\boldsymbol{X}$, $\exists$ a real number δ ($\boldsymbol{x}$) which is the estimate of $\tau(\theta)$ for the set of observations $\boldsymbol{x}$.

To determine an estimate of $\tau(\theta)$, we must specify the form of the loss function associated with $\tau(\theta)$.

Let L (θ, d) is the loss function associated with θ when a decision d (which is a realization of δ ($\boldsymbol{x}$)) is taken. The most obvious form of L (θ, d) is a convex loss function of the form

$$\mathrm{L}(\theta, d) = k(\theta) f_\theta\left(|d - \tau(\theta)|\right)$$

where $k(\theta)$ is a function of θ and f_θ (| d - $\tau(\theta)$ |) is a class of the functions of absolute difference between the estimated value on the basis of d and the function $\tau(\theta)$ to be estimated.

Then d is chosen as to minimize L (θ, d).

We will consider two most commonly used loss functions and will find the best estimates of $\tau(\theta)$ having the specified loss functions.

(i) Let L (θ, d) $= k\left(d - \tau(\theta)\right)^2$ where k is a constant and $\left(d - \tau(\theta)\right)^2$ is the squared error.

This loss function is known as **squared error** loss function. The risk associated with a decision d in this case is

$$r\ (\zeta,\delta\ ) \ = \ kE_\theta\left(\delta(\boldsymbol{x})-\tau(\theta)\right)^2$$

$$= \ k\ \left(\text{MSE of } \delta \text{ at } \theta\right)$$

Thus risk is minimum when $\delta(\boldsymbol{x})$ is MVUE of $\tau(\theta)$ i.e. $\delta(\boldsymbol{x})$ is the mean of the posterior distribution $\zeta_\theta(.)$ of θ.

(ii) Let L (θ, d) $= k\left|d-\tau(\theta)\right|$ where k is a constant and $\left|d-\tau(\theta)\right|$ is the absolute deviation of d about $\tau(\theta)$.

This loss function is known as **mean deviation** loss function. The risk associated with a decision d in this case is

$$r\ (\zeta,\delta\ ) \ = \ kE_\theta\left|\delta(\boldsymbol{x})-\tau(\theta)\right|$$

Thus risk is minimum when $\delta(\boldsymbol{x})$ is MVUE of $\tau(\theta)$ which we know is the median of the posterior distribution $\zeta_\theta(.)$ of θ.

In the examples discussed below, we consider the two most commonly used prior distributions namely the beta distribution and the normal distribution and will obtain the Bayes' estimates of θ.

Example 1: Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution with parameter θ. Then the p.m.f. of $\boldsymbol{X}$ is given by

$$f_\theta\left(x\right)=\begin{cases}\theta^x(1-\theta)^{1-x}\ ; & \text{if } x=0,1\\ 0\ ; & \text{otherwise}\end{cases}$$

Let $\theta\in\Theta$ be a random variable distributed as β (a, b). Then the prior distribution $\xi(\theta)$ of θ is given by

$$\xi_\Theta(\theta)=\begin{cases}\dfrac{1}{\beta(a,b)}\theta^{a-1}(1-\theta)^{b-1}; & 0<\theta<1\\ 0; & \text{otherwise}\end{cases}$$

We want to estimate $\tau(\theta)=\theta$.

Let L (θ, d) $= k\left(d-\tau(\theta)\right)^2$ be the loss function of the decision d.

Now,

$$f_\theta\left(x_1, x_2, ..., x_n\right) = \prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^m(1-\theta)^{n-m}; \quad m=\sum_{i=1}^{n}x_i=0,1,2...n$$

Then the posterior distribution

$$\zeta_\theta(X) = \frac{\zeta(\theta) f_\theta(x)}{\int_\Theta \zeta(\theta) f_\theta(x) d\theta}$$

$$= \frac{\theta^{a+m-1}(1-\theta)^{b+n-m-1}}{\int_0^1 \theta^{a+m-1}(1-\theta)^{b+n-m-1} d\theta}$$

$$= \frac{\theta^{a+m-1}(1-\theta)^{b+n-m-1}}{B(a+m, b+n-m)}$$

Thus $\zeta_\theta(X) \sim B(a+m, b+n-m)$ with mean $\dfrac{a+m}{a+b+n}$ which is the MVUE of $\tau(\theta) = \theta$.

Example 2: Let $X_1, X_2, ..., X_n$ be a random sample from an $N(\theta, \sigma^2)$ distribution where σ^2 is known but θ is a random variable having the density function $N(\mu, \tau^2)$. μ and τ^2 are known. First of all, we find out the posterior distribution of θ.

Now,

$$\zeta_X(\theta) = \frac{\zeta(\theta) f_\theta(x)}{\int_\Theta \zeta(\theta) f_\theta(x) d\theta} \text{ where}$$

$$\zeta(\theta) = \frac{1}{\tau\sqrt{2\pi}} \exp\left\{-\frac{(\theta-\mu)^2}{2\tau^2}\right\}; \text{ and}$$

$$f_\theta(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$$

Therefore $$\zeta(\theta) f_\theta(x) = \frac{1}{\tau\sigma 2\pi} \exp\left\{-\frac{(\theta-\mu)^2}{2\tau^2} - \frac{(x-\theta)^2}{2\sigma^2}\right\}$$

where

$$\frac{(\theta-\mu)^2}{\tau^2} + \frac{(x-\theta)^2}{\sigma^2} = \frac{1}{\tau^2}\left(\theta^2 - 2\theta\mu + \mu^2\right) + \frac{1}{\sigma^2}\left(\theta^2 - 2\theta x + x^2\right)$$

$$= \left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right)\theta^2 - 2\theta\left(\frac{\mu}{\tau^2} + \frac{x}{\sigma^2}\right) + \left(\frac{\mu^2}{\tau^2} + \frac{x^2}{\sigma^2}\right)$$

Let $$\gamma = \frac{1}{\tau^2} + \frac{1}{\sigma^2}$$

Then

$$\frac{(\theta-\mu)^2}{\tau^2}+\frac{(x-\theta)^2}{\sigma^2} = \gamma\left(\theta^2-\frac{2}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\theta+\frac{1}{\gamma^2}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)^2\right)-\frac{1}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)^2 \quad +\left(\frac{\mu^2}{\tau^2}+\frac{x^2}{\sigma^2}\right)$$

$$= \gamma\left(\theta^2-\frac{2}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\theta+\frac{1}{\gamma^2}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)^2\right)-\frac{\tau^2\sigma^2}{\tau^2+\sigma^2}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)^2 \quad +\left(\frac{\mu^2}{\tau^2}+\frac{x^2}{\sigma^2}\right)$$

$$= \gamma\left(\theta-\frac{1}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\right)^2+\left(\frac{(\mu-x)^2}{\tau^2+\sigma^2}\right)$$

$$\Rightarrow \zeta(\theta)f_\theta(x) = \frac{1}{2\pi\tau\sigma}\exp\left\{-\frac{1}{2\gamma}\left(\theta-\frac{1}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\right)^2\right\}\exp\left\{-\left(\frac{(\mu-x)^2}{\tau^2+\sigma^2}\right)\right\}$$

Also $$\int_{-\infty}^{\infty}\zeta(\theta)f_\theta(x)d\theta = \frac{1}{2\pi\sigma\tau}\exp\left\{-\left(\frac{(\mu-x)^2}{\tau^2+\sigma^2}\right)\right\}$$

$$\Rightarrow \quad \zeta_X(\theta) = \sqrt{\frac{\gamma}{2\pi}}\ \exp\left\{-\frac{1}{2\gamma}\left(\theta-\frac{1}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\right)^2\right\}$$

Putting $\mu_X = \frac{1}{\gamma}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)$, we have $\zeta_X(\theta) \sim N\left(\mu_X,\gamma^{-1}\right)$.

Now, let us estimate the Bayes' estimate of θ under squared error loss function.

Let L (θ, d) $= k\left(d-\tau(\theta)\right)^2$ be the loss function of the decision d.

Then we have to determine a so as to minimize EL (θ).

Now,

$$\begin{aligned}
E\text{L}(\theta) &= E\left(k\left(d-\theta\right)^2\right)\\
&= k\,E\left(d-\theta\right)^2\\
&= k\,E\left(d-E(\theta)+E(\theta)-\theta\right)^2\\
&= k\left(E\left(d-E(\theta)\right)^2+2E\left(d-E(\theta)\right)\left(E(\theta)-\theta\right)+E\left(E(\theta)-\theta\right)^2\right)\\
&= k\left(E\left(d-E(\theta)\right)^2+\left(E(\theta)-\theta\right)^2\right)
\end{aligned}$$

as the middle term is 0 and the end term is a constant w.r.t. d.

$\Rightarrow E\mathrm{L}\ (\theta)$ is minimum when $E\left(d - E(\theta)\right)^2$ is minimum, i.e. at $d = E(\theta)$.

Thus in case of quadratic loss function, the best estimate of θ is given by E (θ).

Example 3: During festival seasons, certain products are in heavy demand but once the festival season is over, the demand for such products declines very sharply. For example, during Diwali times, decorative gift items are heavily demanded. Such a dealer has estimated that the demand of such items is a random variable, i.e. it varies but can be estimated with the help of a probability distribution. Denoting the demand by θ, he has obtained the following probability distribution:

$$P(\theta = 1) = 0.01$$

$$P(\theta = 2) = 0.14$$

$$P(\theta = 3) = 0.35$$

$$P(\theta = 4) = 0.5$$

(The values of θ have been chosen for the sake of simplicity). The order for manufacturing of such items has to be placed in advance. For each item sold during the festival period, the dealer makes a profit of Rs. 1500 but once the festival period is over, he has to bear a loss of Rs. 1000 as the item may have to be sold at a discount or may not be sold at all. How many units of the product should he order for manufacturing so as to minimize the loss?

Sol: Let the decision of the dealer be denoted by d. Then the loss function of the dealer is given by

$$L(d,\theta) = \begin{cases} 1500(\theta - d); & \text{if } d < \theta \\ 0; & \text{if } d = \theta \\ 1000(d - \theta); & \text{if } d > \theta \end{cases}$$

True value of θ is unknown but the optimal value of d will be the point estimate of θ. We prepare the loss table of the dealer and the decision corresponding to which the loss is minimum, will be the required value.

Table 2.1

θ / d	1 $P(\theta=1)=0.01$	2 $P(\theta=2)=0.14$	3 $P(\theta=3)=0.35$	4 $P(\theta=4)=0.5$	$EL(d)$
1	0	1500	3000	4500	3510
2	1000	0	1500	3000	2035
3	2000	1000	0	1500	910
4	3000	2000	1000	0	**660**

Thus he should place an order of 4 units in order to minimize his expected loss.

Further suppose that a market research has prompted him to choose the following probability distribution in place of the one which he was using earlier:

$$P(\theta=1)=0.05$$

$$P(\theta=2)=0.15$$

$$P(\theta=3)=0.45$$

$$P(\theta=4)=0.35$$

Then we shall see how the decision is being affected.

Table 2.2

θ / d	1 $P(\theta=1)=0.05$	2 $P(\theta=2)=0.15$	3 $P(\theta=3)=0.45$	4 $P(\theta=4)=0.35$	$EL(d)$
1	0	1500	3000	4500	3150
2	1000	0	1500	3000	1775
3	2000	1000	0	1500	**775**
4	3000	2000	1000	0	900

Now the optimal decision is to order 3 units of the product in place of 4 (earlier decision).

2.4 Linear pay-off functions: The two action problem

The tabular form representation is convenient if the state space of the nature is discrete. But in case of continuous space of states of nature, where the uncertainty is defined over uncountably infinite points, it may not be possible to represent the pay-offs in tabular form. In such cases, the pay-offs are represented in the form of a function. It may be assumed that the functional form of such functions is linear in terms of the states of nature. Mathematically such a pay-off function may be defined as

$$F(d,\theta) = \alpha + \beta\theta; \quad \theta \in \Theta$$

where α and β are constants.

Linear pay-off functions are used widely in decision making problems. We will consider the case where there are two possible decisions or actions for a continuous state space.

Let the two possible outcomes are d_1 and d_2. Then the corresponding pay-off functions are

$$F(d_1,\theta) = \alpha_1 + \beta_1\theta$$

$$F(d_2,\theta) = \alpha_2 + \beta_2\theta \ ; \ \theta \in \Theta$$

Let $\beta_2 > \beta_1$. We want to find the conditions under which it would be optimal to use d_1 for the prior distribution $\xi(.)$ of θ.

Obviously, it will be optimal to use d_1 if (EV denotes the expected value of a decision)

$$EV(d_1) > EV(d_2)$$

or

$$\alpha_1 + \beta_1 E(\Theta) \ > \ \alpha_2 + \beta_2 E(\Theta)$$

$$\Leftrightarrow \quad \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \ > \ E(\Theta)$$

This is the condition needed in order for d_1 to be optimal. Reversing the inequality, we get condition for d_2 to be optimal. In case of equality, the user is indifferent to use of decisions d_1 or d_2. The corresponding value of θ is called the **break even value** of θ.

The break even value (say, θ_b) of $\theta \in \Theta$ can be used to decide the optimal decision. If $EV(d_1) > \theta_b$ use d_1, else use d_2.

It is evident from the above discussion that the only requirement from $\xi(\theta)$ is that of $E(\theta)$.

Consider again the linear pay-off functions associated with the two action problem

$$F(d_1,\theta) = \alpha_1 + \beta_1\theta$$

$$F(d_2,\theta) = \alpha_2 + \beta_2\theta \;\; ; \;\; \theta \in \Theta$$

such that $\beta_2 > \beta_1$. The condition implies that the slope of the line segment corresponding to d_2 is higher than the slope of the line segment corresponding to the decision d_1 and this is true for all values of θ in Θ. This means that the slope of the line segment corresponding to the decision d_1 should either be negative or should be as follows which implies that for some values of θ in Θ, $\beta_2 > \beta_1$ and for remaining values $\beta_2 < \beta_1$

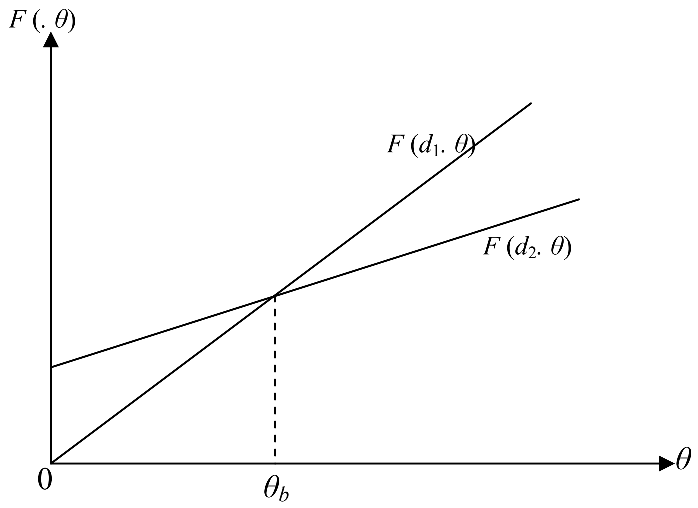

Fig. 2.1

For negative slope

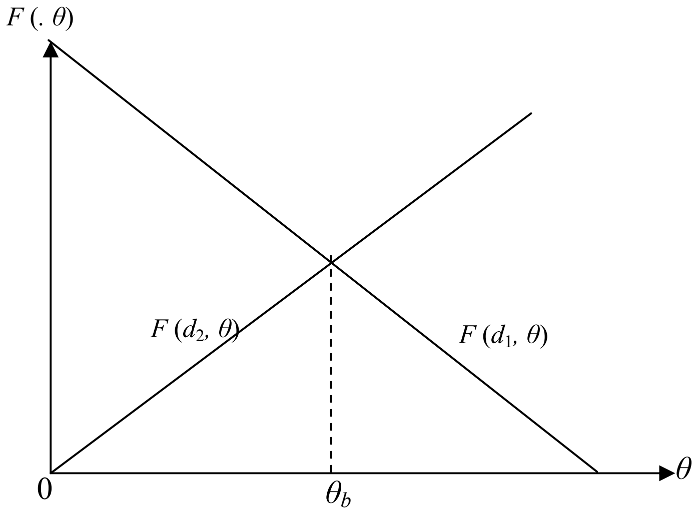

Fig. 2.2

The graph indicates that on left of θ_b, $F(d_1,\theta)$ is higher than $F(d_2,\theta)$ and on the right side of θ_b, $F(d_2,\theta)$ is higher than $F(d_1,\theta)$. We choose that action, which on average is best. Then the decision rule may be stated as:

If $E(\Theta) < \theta_b$, choose the action (decision) d_1 as the best action (decision). If $E(\Theta) > \theta_b$, choose the action (decision) d_2 as the best action (decision).

The following examples will illustrate the point.

Example 4: A sales man has been offered two salary plans. He may work on commission basis receiving a commission of 10% on the sales he executes or on salary basis receiving a fixed salary of Rs. 5000 per month plus a 5% commission on the sales he executes. If he wishes, he can switch over to the other plan (than the one which he has opted for) at the end of the year. The total sales executed by the salesman can be assumed to be a continuous random variable. What should be the decision of the salesman?

Sol: Let the monthly sales be denoted by the variable θ. Then

$$F(d_1,\theta) = 0.10\theta$$

$$F(d_2,\theta) = 5000 + 0.5\theta \;\; ; \;\; \theta \in \Theta$$

First plan is optimal if

$$F(d_1,\theta) > F(d_2,\theta)$$

$$\Leftrightarrow 0.10\theta > 5000 + 0.5\theta$$

$$\Leftrightarrow 0.05\theta > 5000$$

$$\Leftrightarrow \theta > 100000$$

i.e., if monthly sales are more than Rs. 1,00,000, it is advisable to go for the first plan otherwise it is better to go for the second plan.

The break-even point is θ = Rs. 1, 00,000 when the pay-off is Rs. 10,000.

Graphically,

$F(.\ \theta)$ ('000)

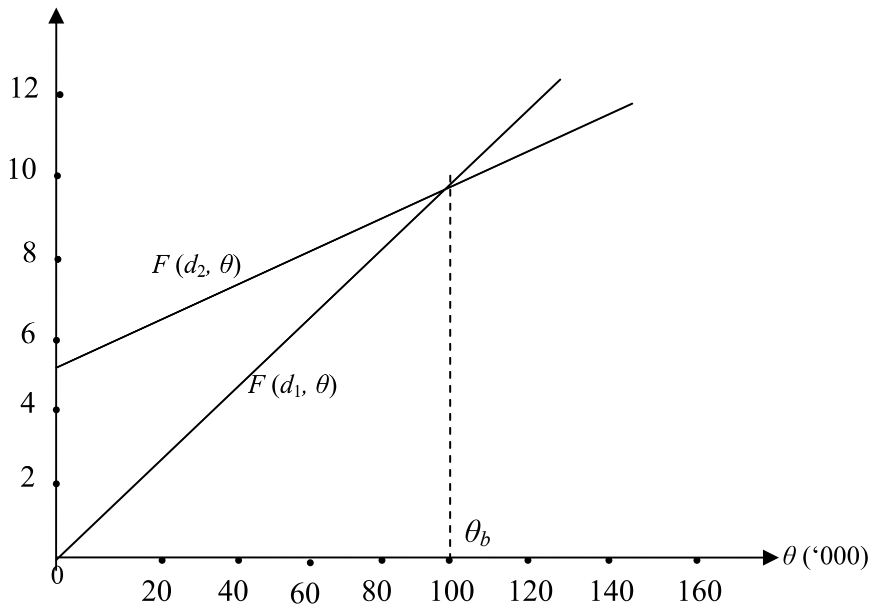

Fig. 2.3

If $E\ (\theta) > 1,\ 00{,}000$, then the salesman should work on commission basis and if the expected sates are less than Rs. 1, 00,000, he should work on salary basis.

Opportunity loss functions and linear pay-off functions

In a two action problem if the pay-off functions of both the actions are linear (w. r. to the state space), then the corresponding loss functions are also linear. The opportunity loss functions, as in the case of pay-off functions, are functions of the state space Θ. The opportunity loss is zero in case of the optimal decision so the function takes value 0 on one side of the break even point; on the other side of the break-even point; it would be a straight line. For the linear pay-off functions considered above, the opportunity loss functions can be represented as follows:

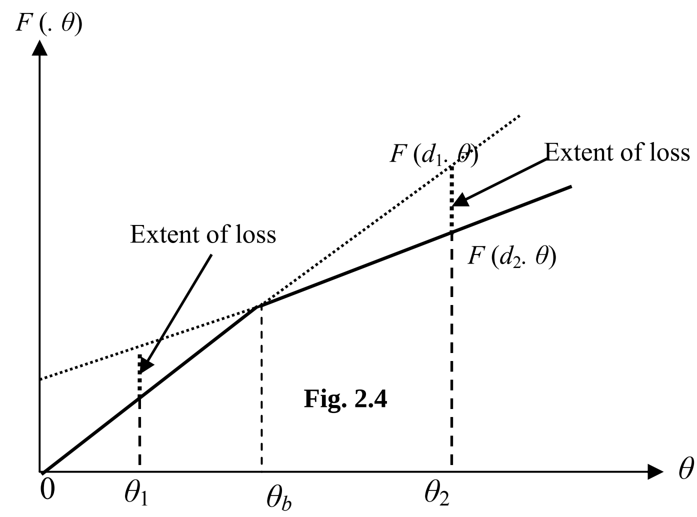

Fig. 2.4

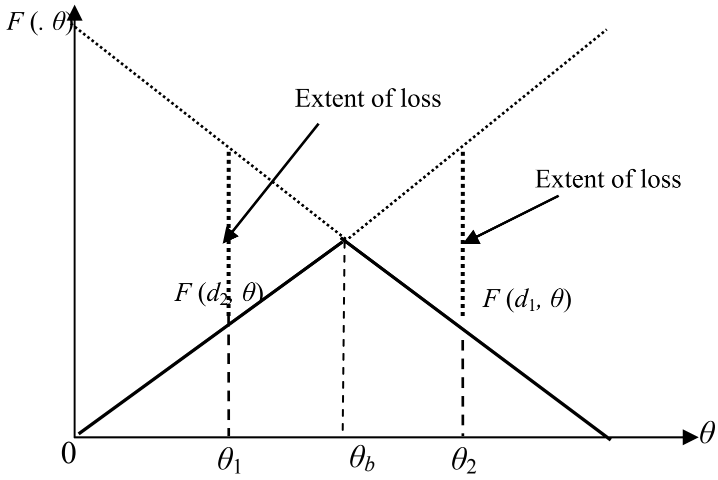

Fig. 2.5

In Fig. 2.5, consider the V-shaped region marked by the dotted lines. This is the pay-off function of the best action that can be taken at any point of time for any value θ in Θ. Such a function is called **piece-wise linear**. For any action (decision) d_j, the extent of loss suffered for any value θ in Θ is the region between the lines representing $F(d_j, \theta)$ and the piece wise linear function representing the pay-off of the best action. Let $j = 1$. Then the opportunity loss can mathematically be represented as

$$OL(d_1,\theta) = \begin{cases} F(d_1,\theta) - F(d_2,\theta); & \text{if} \quad \theta < \theta_b \\ F(d_2,\theta) - F(d_1,\theta); & \text{if} \quad \theta > \theta_b \end{cases}$$

This, in turn, can be written as

$$OL(d_1,\theta) = \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ \alpha_2 - \alpha_1 + (\beta_2 - \beta_1)\theta; & \text{if} \quad \theta > \theta_b \end{cases}$$

$$= \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ (\beta_2 - \beta_1)(\theta - \theta_b); & \text{if} \quad \theta > \theta_b \end{cases}$$

Graphically

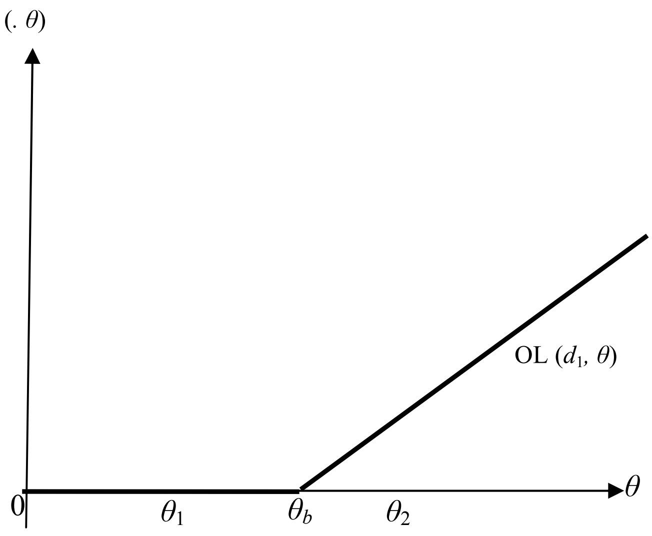

Fig. 2.6

Similarly, if $j = 2$, then

$$OL(d_2, \theta) = \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ (\beta_1 - \beta_2)(\theta - \theta_b); & \text{if} \quad \theta > \theta_b \end{cases}$$

and graphically

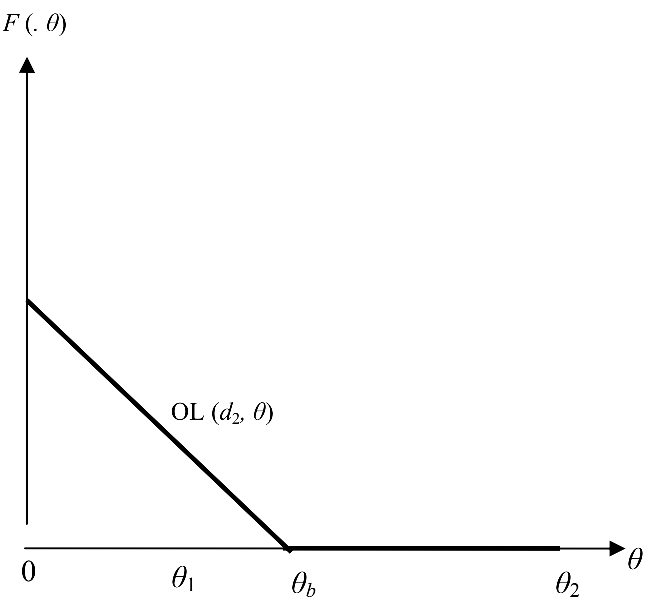

Fig. 2.7

Now, we can compute the expected opportunity loss of the decision problem.

$$EOL(d_1) = \int_{-\infty}^{\infty} OL(d_1,\theta)\xi(\theta)d\theta$$

$$= \int_{-\infty}^{\theta_b} 0.\xi(\theta)d\theta + \int_{\theta_b}^{\infty} \left(\alpha_2 - \alpha_1 + (\beta_2 - \beta_1)\theta\right)\xi(\theta)d\theta$$

$$= \int_{\theta_b}^{\infty} \left(\alpha_2 - \alpha_1 + (\beta_2 - \beta_1)\theta\right)\xi(\theta)d\theta$$

$$= (\beta_2 - \beta_1)\int_{\theta_b}^{\infty} \left(\theta - \theta_b\right)\xi(\theta)d\theta$$

Similarly,

$$EOL(d_2) = \int_{-\infty}^{\infty} OL(d_2,\theta)\xi(\theta)d\theta$$

$$= \int_{-\infty}^{\theta_b} \left(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)\theta\right)\xi(\theta)d\theta + \int_{\theta_b}^{\infty} 0.\xi(\theta)d\theta$$

$$= \int_{-\infty}^{\theta_b} \left(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)\theta\right)\xi(\theta)d\theta$$

$$= (\beta_1 - \beta_2)\int_{-\infty}^{\theta_b} \left(\theta - \theta_b\right)\xi(\theta)d\theta$$

Therefore, $\qquad EVPI \;=\; \min\left(EOL(d_1), EOL(d_2)\right)$

Linear pay-off function and beta distribution

Consider the opportunity loss functions associated with the decisions d_1 and d_2

$$OL(d_1,\theta) = \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ \alpha_2 - \alpha_1 + (\beta_2 - \beta_1)\theta; & \text{if} \quad \theta > \theta_b \end{cases}$$

$$= \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ \gamma_1 + \delta_1\theta \;\; \text{(say)}\,; & \text{if} \quad \theta > \theta_b \end{cases}$$

and

$$OL(d_2,\theta) = \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ \alpha_1 - \alpha_2 + (\beta_1 - \beta_2)\theta; & \text{if} \quad \theta > \theta_b \end{cases}$$

$$= \begin{cases} 0; & \text{if} \quad \theta \le \theta_b \\ \gamma_2 + \delta_2\theta \ \text{(say)}\ ; & \text{if} \quad \theta > \theta_b \end{cases}$$

Let $\Theta \sim \beta 1\ (a, b)$. Then the prior distribution $\xi(\theta)$ of θ is given by

$$\xi_\Theta(\theta) = \begin{cases} \dfrac{1}{\beta(b+1, a-b+1)}\theta^b (1-\theta)^{a-b}; & 0 < \theta < 1 \\ 0; & \text{otherwise} \end{cases}$$

$$= \begin{cases} (a+1)\dbinom{a}{b}\theta^b (1-\theta)^{a-b}; & 0 < \theta < 1 \\ 0; & \text{otherwise} \end{cases}$$

The mean and the variance of the distribution are given by

$$E(\theta) = \frac{b+1}{a+2}; \qquad \text{var}(\theta) = \frac{ab+a+1-a^2}{(a+2)^2(a+3)}$$

Let a and b be non-negative integers. Further let the population from which the observations $\boldsymbol{X}$ have been taken be distributed as *Bin* (n, θ). Then the probability mass function of $\boldsymbol{X}$ is given by

$$P(X = k) = \begin{cases} \dbinom{n}{k}\theta^k (1-\theta)^{n-k}; & 0 < \theta < 1;\ k = 0,1,2,\dots,n \\ 0; & \text{otherwise} \end{cases}$$

The mean and the variance of the distribution are given by

$$E(X) = n\theta; \qquad \text{var}(\theta) = n\theta(1-\theta)$$

Then we have the following result:

Result 1: The relationship between the cumulative density function of a beta variable and the cumulative mass function of a binomial variable is given by

$$\int_0^{\theta_b} \xi(\theta) \;=\; P(X \geq b+1 \mid n = a+1,\; \theta \;=\; \theta_b)$$

$$= \sum_{k=b+1}^{a+1} \binom{n}{k} \theta_b^{\,k} (1-\theta_b)^{n-k}$$

Now, we shall calculate the expected opportunity loss for a decision.

$$EOL(d_2) = \int_0^{\theta_b} \left(\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)\theta\right) \xi(\theta) d\theta$$

$$= \int_0^{\theta_b} \left(\gamma_2 + \delta_2 \theta\right) \xi(\theta) d\theta$$

$$= \gamma_2 P(X \geq b+1 \mid n = a+1,\; \theta \;=\; \theta_b) + \;\delta_2 \int_0^{\theta_b} \theta \; \xi(\theta) d\theta$$

where

$$\int_0^{\theta_b} \theta \; \xi(\theta) d\theta \;\;=\;\; \int_0^{\theta_b} \theta \; \left(\frac{(a+1)!}{b!(a\text{-}b)!}\right) \theta^b (1-\theta)^{a-b} \, d\theta$$

$$= \left(\frac{b+1}{a+2}\right) \int_0^{\theta_b} \left(\frac{a+2}{b+1}\right) \left(\frac{(a+1)!}{b!(a\text{-}b)!}\right) \theta^{b+1} (1\text{-}\theta)^{a\text{-}b} \, d\theta$$

$$= \left(\frac{b+1}{a+2}\right) \int_0^{\theta_b} \left(a+2\right) \left(\frac{(a+1)!}{(b+1)!}\right) \theta^{b+1} (1\text{-}\theta)^{a\text{-}b} \, d\theta$$

$$= \left(\frac{b+1}{a+2}\right) \int_0^{\theta_b} pdf \text{ of a beta variable with parameters } (a+1) \text{ and } (b+1)$$

$$= \left(\frac{b+1}{a+2}\right) P(X \geq b+2 \mid n = a+2, \theta = \theta_b)$$

$$\therefore \;\; EOL(d_2) = \gamma_2 P(X \geq b+1 \mid n = a+1,\; \theta \;=\; \theta_b) + \;\delta_2 \left(\frac{b+1}{a+2}\right) P(X \geq b+2 \mid n = a+2, \theta = \theta_b)$$

Also,

$$EOL(d_1) = \int_{\theta_b}^{1} \left(\gamma_1 + \delta_1 \theta\right) \xi(\theta) d\theta$$

$$= \gamma_1 \int_{\theta_b}^{1} \xi(\theta) d\theta + \delta_1 \int_{\theta_b}^{1} \theta \xi(\theta) d\theta$$

$$= \gamma_1 \left(1 - \int_0^{\theta_b} \xi(\theta) d\theta \right) + \delta_1 \left(\int_0^1 \theta \xi(\theta) d\theta - \int_0^{\theta_b} \theta \xi(\theta) d\theta \right)$$

$$= \gamma_1 \left(1 - P(X \geq b+1 \mid n = a+1,\ \theta\ =\ \theta_b) \right)$$

$$+ \delta_1 \left(\left(\frac{b+1}{a+2} \right) - \left(\frac{b+1}{a+2} \right) P(X \geq b+2 \mid n = a+2,\ \theta\ =\ \theta_b) \right)$$

$$= \gamma_1 P(X < b+1 \mid n = a+1,\ \theta\ =\ \theta_b) + \delta_1 \left(\frac{b+1}{a+2} \right) P(X < b+2 \mid n = a+2,\ \theta\ =\ \theta_b)$$

The EOL of the two actions can be computed using the cumulative binomial tables.

Example 6: In consumables' industry, packaging is an essential process of production. In such a unit, packaging is done with the help of a machine for which probability of an improper packing is θ. During a single operation of the machine 5,000 units are packed after which the machine is cleaned and re adjusted if needed. This means that θ varies from run to run and hence is a random variable. There are two alternatives before the production manager. After each cleaning, the machine may be used without any inspection. In this case, if any unit is found improperly packed during the quality control inspection, it has to be replaced by a new unit. The cost of this operation is Rs. 2 per unit. If the manager goes for inspection of the machine by an inspector before reusing it, the probability of any defective in the new run is practically zero. But the cost of the time lost in the operation is Rs. 500. What should be the decision of the production manager if θ follows a beta distribution with parameters a = 14 and b = 0?

Sol: Define the actions:

d_1: Call the expert to inspect the machine

d_2: Do not call the expert and use the machine as it is after cleaning.

Let the cost function of the decisions is denoted by $C(d, \theta)$. Then

$$C(d_1, \theta) = 500$$

$$C(d_2, \theta) = 2 \times 5000 \times \theta$$

$$= 10000\theta$$

Now, we will locate the break even value θ_b of θ.

$$C(d_1,\theta_b) = C(d_2,\theta_b)$$

$$\Leftrightarrow \; 500 = 10000\theta$$

$$\Leftrightarrow \; \theta = 0.05$$

Graphically

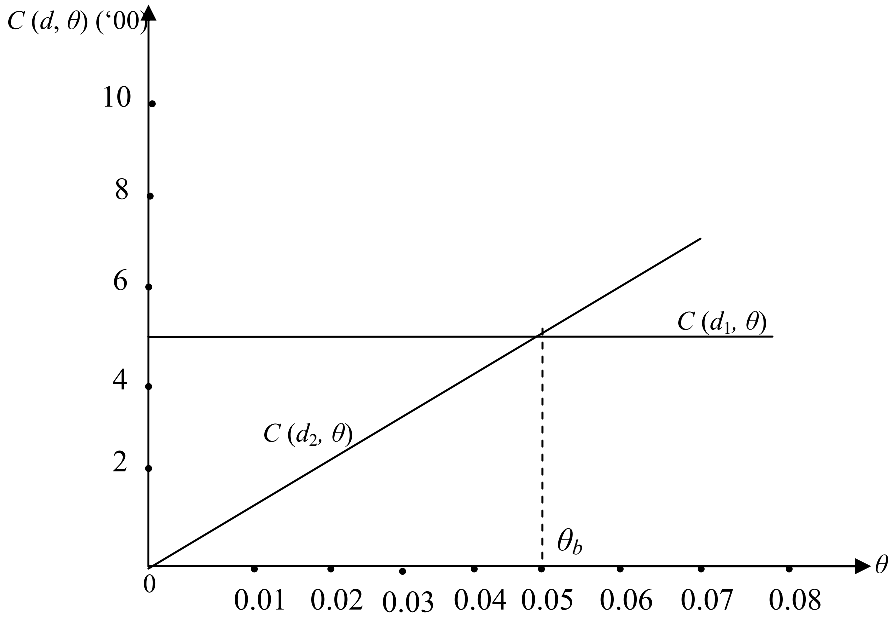

Fig. 2.8

The opportunity loss functions of the two decisions are given by

$$OL(d_1,\theta) = \begin{cases} 500 - 10000\theta; & if \;\; \theta < \theta_b \\ 0; & if \;\; \theta \geq \theta_b \end{cases}$$

and

$$OL(d_2,\theta) = \begin{cases} 10000\theta - 500; & if \;\; \theta \leq \theta_b \\ 0; & if \;\; \theta > \theta_b \end{cases}$$

Graphically, the functions can be represented as

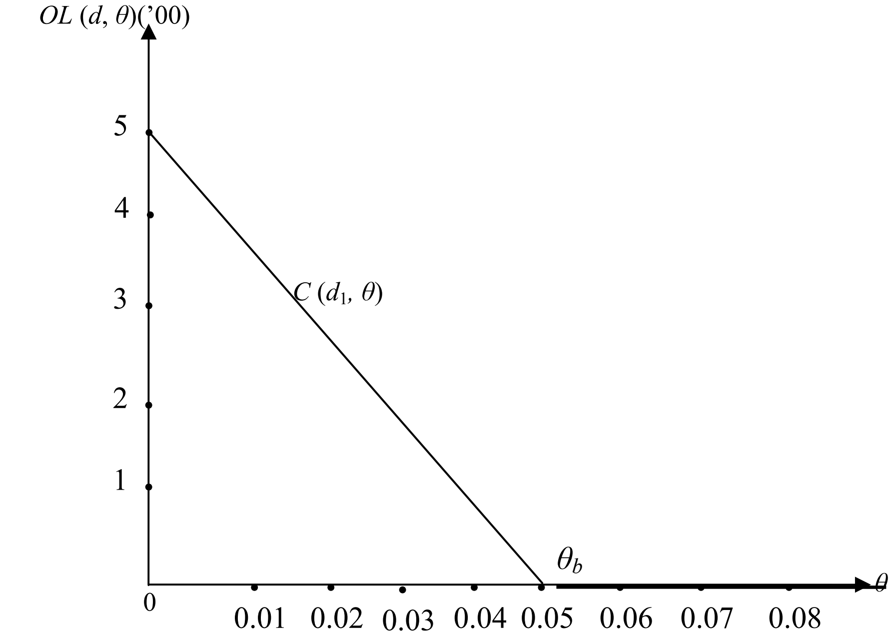

Fig. 2.9

And

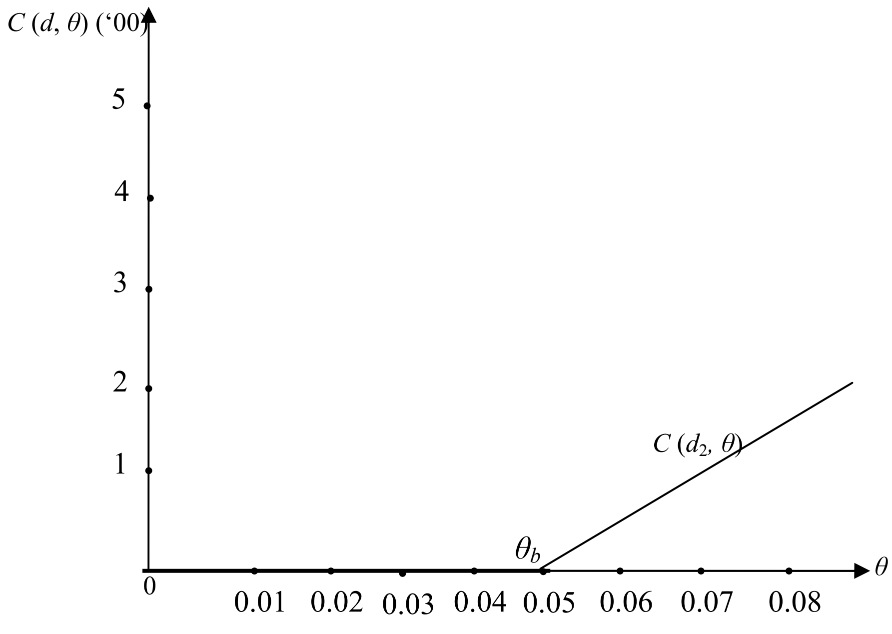

Fig. 2.10

The expected opportunity loss is given by

$$EOL(d_1) \;=\; 500P(X \geq 1 \mid n=15, \theta=0.05) \;-\; 10000\left(\frac{1}{16}\right)P(X \geq 2 \mid n=16, \theta=0.05)$$

$$= \;500(0.5367) - \frac{10000}{16}(0.1892)$$

$$= \;268.35\text{-}118.25$$

$$= \;150.1$$

and

$$EOL(d_2) \;=\; -500P(X < 1 \mid n=15, \theta=0.05) \;+\; 10000\left(\frac{1}{16}\right)P(X < 2 \mid n=16, \theta=0.05)$$

$$= \;-500(0.4633) + \frac{10000}{16}(0.8108)$$

$$= \;506.75\text{-}231.65$$

$$= \;293.1$$

As $EOL\ (d_1) < EOL\ (d_2)$ hence first decision should be taken, i.e. expert should be called to inspect the machine after each cleaning.

Linear loss functions and the normal distribution

Again consider the two action problem with the decisions d_1 and d_2 and the opportunity loss functions

$$OL(d_1, \theta) \;=\; \begin{cases} 0; & \text{if} \quad \theta \leq \theta_b \\ \delta_1(\theta - \theta_b)\ ; & \text{if} \quad \theta > \theta_b \end{cases}$$

and

$$OL(d_2, \theta) \;=\; \begin{cases} 0; & \text{if} \quad \theta < \theta_b \\ \delta_1(\theta_b - \theta)\ ; & \text{if} \quad \theta \geq \theta_b \end{cases}$$

Further let $\Theta \sim N\,(\mu, \sigma^2)$. Thus μ is the expected value of the parameter θ. Two cases arise:

Case 1: $\theta_b > \mu$.

$$EOL(d_1) = \delta_1 \int_{\theta_b}^{\infty} (\theta - \theta_b)\xi(\theta) d\theta$$

$$= \frac{\delta_1}{\sigma\sqrt{2\pi}} \int_{\theta_b}^{\infty} (\theta - \theta_b) e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2} d\theta$$

Let $Z = \dfrac{\theta\text{-}\mu}{\sigma} \Rightarrow dZ = \dfrac{d\theta}{\sigma}$

and $\theta = \sigma Z + \mu$

$$\Rightarrow \quad EOL(d_1) = \frac{\delta_1}{\sqrt{2\pi}} \int_{\left(\frac{\theta_b-\mu}{\sigma}\right)}^{\infty} \left(\sigma Z - (\theta_b - \mu)\right) e^{-\frac{1}{2}Z^2} dZ; \quad \theta_b > \mu$$

$$= \frac{\delta_1}{\sqrt{2\pi}} \sigma \int_{\left(\frac{\theta_b-\mu}{\sigma}\right)}^{\infty} Z e^{-\frac{1}{2}Z^2} dZ - \frac{(\theta_b - \mu)}{\sqrt{2\pi}} \int_{\left(\frac{\theta_b-\mu}{\sigma}\right)}^{\infty} e^{-\frac{1}{2}Z^2} dZ$$

$$= \delta_1 \sigma \left(\frac{e^{-\frac{1}{2}\left(\frac{\theta_b-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}} - \left(\frac{\theta_b - \mu}{\sigma}\right) P\left(Z > \frac{\theta_b - \mu}{\sigma}\right) \right)$$

Denote by $D = \left(\dfrac{\theta_b - \mu}{\sigma}\right)$ the unit normal loss deviation and by $L_N(D) = \dfrac{e^{-\frac{1}{2}D^2}}{\sqrt{2\pi}} - DP(Z > D)$, the unit normal loss function. Then

$$EOL(d_1) = \delta_1 \sigma L_N(D)$$

where $L_N(D)$ can be obtained from the tables.

Again,

$$EOL(d_2) = \delta_2 \int_{-\infty}^{\theta_b} (\theta_b - \mu)\xi(\theta) d\theta; \quad \theta_b > \mu$$

$$= \frac{\delta_2}{\sqrt{2\pi}} \sigma \int_{-\infty}^{\theta_b} (\theta_b - \mu) e^{-\frac{1}{2}\left(\frac{\theta-\mu}{\sigma}\right)^2} d\theta$$

Let $Z = \dfrac{\theta\text{-}\mu}{\sigma} \Rightarrow dZ = \dfrac{d\theta}{\sigma}$

$$\Rightarrow \quad EOL(d_2) = \frac{\delta_2}{\sqrt{2\pi}} (\theta_b - \mu) \int_{-\infty}^{\theta_b} e^{-\frac{1}{2}Z^2} dZ - \sigma \frac{\delta_2}{\sqrt{2\pi}} \int_{-\infty}^{\theta_b} Z e^{-\frac{1}{2}Z^2} dZ$$

$$= \delta_2(\theta_b-\mu)P\left(Z \le \frac{\theta_b-\mu}{\sigma}\right) - \frac{\delta_2}{\sqrt{2\pi}}\sigma \int_{-\infty}^{\left(\frac{\theta_b-\mu}{\sigma}\right)} e^{-\frac{1}{2}Z^2}\,dZ$$

$$= \delta_2(\theta_b-\mu)\left(1-P\left(Z > \frac{\theta_b-\mu}{\sigma}\right)\right) - \frac{\sigma\delta_2}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta_b-\mu}{\sigma}\right)^2}$$

$$= \sigma\delta_2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta_b-\mu}{\sigma}\right)^2} - \frac{\theta_b-\mu}{\sigma} P\left(Z > \frac{\theta_b-\mu}{\sigma}\right) + \left(\frac{\theta_b-\mu}{\sigma}\right)\right)$$

$$= \sigma\delta_2\left(L_N(D)+D\right)$$

Graphically, the two loss functions are mirror image of each other.

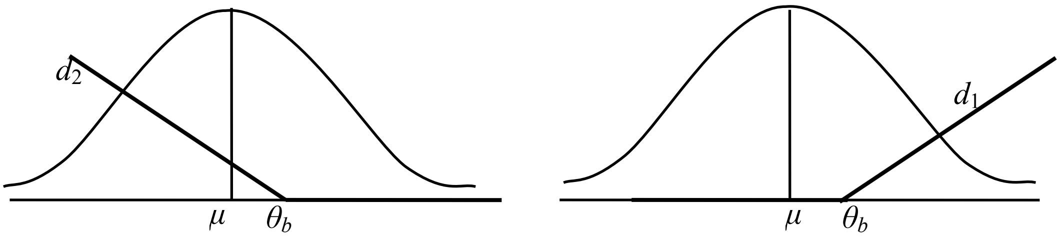

Fig. 2.11

Case 2: $\theta_b < \mu$.

The opportunity loss functions in this case are given as follows:

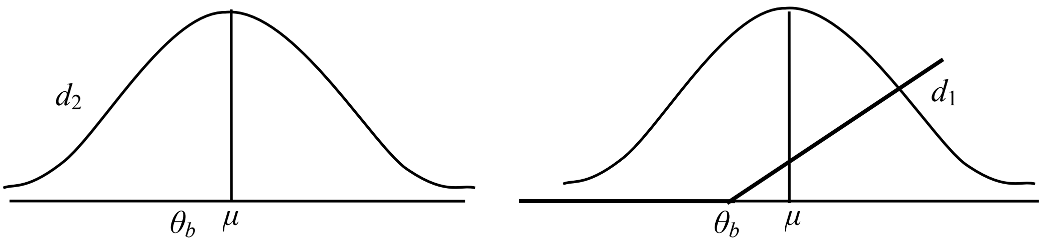

Fig. 2.12

Also

$$EOL(d_2) = \delta_2\sigma L_N(D)$$ (In this case, the OL line does not cross the mean)

and

$$EOL(d_1) = \delta_1\sigma\left(L_N(D)+D\right)$$ (In this case, the OL line crosses the mean)

We compute the EOL of the two decisions on the basis of E (Θ) versus θ_b and obtain the optimal EOL. This EOL is EVPI also.

Example 7: A producer wants to introduce a new variant of his product in the market. The fixed cost of this decision is Rs. 1, 00,000 and every unit of the new variant will fetch a profit of Rs. 2.50 per unit. If he does not introduce the product, he will not earn or loose anything. The producer, however, is uncertain about the number of unit which he can sell in the market and associates his expectations with a normal distribution with mean 50,000 and standard deviation 25,000. What should be the decision of the producer? How much do you think is he willing to spend to get exact information about his future sales?

Sol: Let θ be the random variable denoting the sale of the new variant. Then

$$\theta \sim N\,(50{,}000,\ 625(10)^6)$$

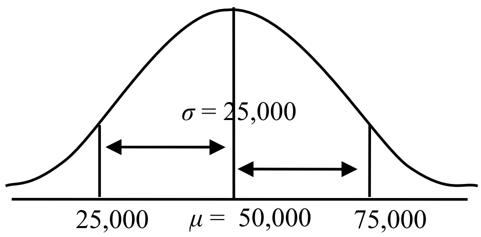

Fig. 2.13

We define the two actions of the producer, along with their pay-offs as follows:

d_1: Launch the new variant

$$P(d_1,\theta) = -1{,}00{,}000 + (2.50)\theta$$

d_2: Do not launch the new variant.

$$P(d_2,\theta) = 0$$

The break-even point is then given by

$$P(d_1,\theta) = P(d_2,\theta)$$

$$\Leftrightarrow \quad -1{,}00{,}000 + (2.50)\theta = 0$$

$$\Leftrightarrow \quad \theta_b = 40{,}000 < 50{,}000 = \mu$$

The expected pay-offs are given as

$$EP(d_1) = -1{,}00{,}000 + (2.50)80{,}000 = Rs.\ 1{,}00{,}000$$

and

$$EP(d_1) = 0$$

With this information, the producer should go for the new variant of his product.

The opportunity loss functions of the two decisions are given by

$$OL(d_1,\theta) \quad = \begin{cases} (2.50)(40,000-\theta) & \text{if } \theta < 40,000 \\ 0 & \text{if } \theta \geq 40,000 \end{cases}$$

and

$$OL(d_2,\theta) \quad = \begin{cases} (2.50)(\theta-40,000) & \text{if } \theta > 40,000 \\ 0 & \text{if } \theta \leq 40,000 \end{cases}$$

Then EVPI can be calculated as

$$\begin{aligned} EVPI &= \delta_2 \sigma L_N(D) \\ &= (2.50)(50{,}000) L_N\left(\frac{40,000-50000}{25000}\right) \\ &= (2.50)(50{,}000) L_N(0.40) \\ &= (2.50)(50{,}000)(0.2304) \\ &= 28800 \end{aligned}$$

This is the amount that the producer is willing to spend to get more exact information about his future sales.

Example 8: The cost of production of a certain item is Rs. 10, 00,000 out of which Rs. 4, 00,000 is the fixed cost and Rs. 10 per unit is the variable cost for every unit produced. The product can be sold for Rs. 20 per unit. However the sales manager is not sure of the exact number of units that he can sell. He estimates it with a normal distribution with parameters (50,000 and 25,000).

(a) What should be the decision of the producer about the number of units to be produced?

(b) What should be the minimum number of units sold for the production to continue?

Sol:

Define the actions:

d_1: Launch the product.

d_2: Do not launch the product.

Let X be the random variable denoting the level of sales made by the producer.

(a) The profit per unit of the producer is Rs. 10 so that the profit function can be written as

$$P(d_1, X) = -10,00,000 + 10X$$

$$P(d_2, X) = 0$$

Then expected profits are given by

$$EP(d_1) = -1000000 + 10(50000)$$

$$= -500000$$

$$EP(d_2) = 0$$

Obviously the optimal action is not to go for production.

(b) The break-even point is given by

$$10,00,000 = 10X$$

$$\Rightarrow X = 1,00,000$$

If the producer can ensure a sale of minimum 1, 00,000 units, only then should he continue with the production.

Problems

1. A medical research firm is planning to introduce a new drug to tackle high blood pressure. However due to intense competition in the market, it is not very sure about its success and it identifies the proportion p of market it would be able to capture with the following probability distribution:

$$P(p) = \begin{cases} 0.05\,; & \text{if } p = 0.10 \\ 0.08; & \text{if } p = 0.12 \\ 0.13; & \text{if } p = 0.14 \\ 0.30; & \text{if } p = 0.15 \\ 0.34; & \text{if } p = 0.16 \\ 0.10; & \text{if } p = 0.20 \end{cases}$$

   What are the mean and variance of p? If the level of market's profit is Rs. 10 million, what is the expected profit of the firm?

2. A textile company is planning to introduce a new fabric in the market for which it is contemplating an advertisement campaign at a cost of Rs. 50, 00,000. If campaign is successful, the estimate of the firm is that the sales of the firm will grow by Rs. 10, 00, 00 000 thus increasing the profits of the firm by 20%. However they are very sure about the level of sales and identify it with a normal distribution with parameters (10, 00, 00 000, 2, 00, 00 000). The two actions that the company can take are (i) to advertise; (ii) not to advertise. What do you think should be the optimal action? What is EVPI?

3. A company has received offers from two different companies about the machines producing same type of product. The details of the two machines are as follows:

Machine A:	Cost price:	Rs. 10, 00,000
	Running cost:	Rs. 10 per unit.
Machine B:	Cost price:	Rs. 25, 00,000
	Running cost:	Rs. 3 per unit.

   The quality of the product on the two machines is same. Also there is not substantial difference in the run time of the two machines.

   (a) Determine the level of sales at which the company would be indifferent to the purchase of the two machines.

   (b) The annual levels of the sales are assumed to follow a normal distribution with mean 80,000 and standard deviation 40,000. Which machine should be purchased with this information?

(c) Find the expected value of the perfect information.

4. A toy manufacturing company believes that the number of units of a new toy which they are planning to introduce in the market will follow a normal distribution with parameters (200000, 50000). Every unit of the toy will fetch them a profit of Rs. 3. But the cost associated with the introduction of the toy is Rs. 1000000. Should they introduce the toy ot not? What is the expected value of perfect information?

5. Consider a variable p which is distributed as a beta variable with parameters $\alpha = 3$ and $\beta = 25$. The pay-offs associated with the two actions a_1 and $a2$ are as follows:

$$P(a_1, p) = 50p$$

$$P(a_2, p) = -50 + 200p$$

What is the break-even point for p. Write the opportunity loss functions and draw the graph of the same. Which action do you think is optimal? Find EVPI.

6. An institution needs printer refills for computer printers. It has obtained offers from two different vendors. Vendor A will supply the refills at a cost of Rs. 900 per unit and will replace any defective refills free of cost. Vendor B will supply the refills at a cost of Rs. 750 but will replace any defective refills at a cost of Rs. 500 per unit. The defective refills follow a beta distribution with parameters $\alpha = 3$ and $\beta = 30$. What should be the decision of the institution? How would the decision change if

(a) $\alpha = 5$ and $\beta = 20$

(b) $\alpha = 3$ and $\beta = 30$

7. A sales man has been offered two salary plans. He may work on commission basis receiving a commission of 15% on the sales he executes or on salary basis receiving a fixed salary of Rs. 10000 per month plus a 5% commission on the sales he executes. If he wishes, he can switch over to the other plan (than the one which he has opted for) at the end of the year. The total sales executed by the salesman can be assumed to be a normal variable with mean 55,000 and s.d. 2500. What should be the decision of the salesman?

8. Consider the following pay-off functions:

$$P(a_1, \theta) = 100 + 50\theta$$

$$P(a_2, \theta) = -150 + 75\theta$$

where θ is a normal variable with

(i) mean = 5 and s.d. = 3

(ii) mean = 10 and s.d. = 4

(iii) mean = 8 and s.d. = 10

In all the three cases, choose the optimal action. Find EVPI that $\theta = 7.5$.

9. In the above problem, choose the optimal action if θ is a beta variable with

(a) $\alpha = 5$ and $\beta = 30$

(b) $\alpha = 10$ and $\beta = 4$

(c) $\alpha = 8$ and $\beta = 10$

10. An automobile dealer has estimated that the monthly demand of cars during the end months of the year is a random variable, i.e. it varies but can be estimated with the help of a probability distribution. Denoting the demand by θ, he has obtained the following probability distribution:

$$P(\theta = 5) = 0.05$$

$$P(\theta = 6) = 0.14$$

$$P(\theta = 7) = 0.55$$

$$P(\theta = 8) = 0.26$$

(The values of θ have been chosen for the sake of simplicity). The order for cars has to be placed in advance. For each car sold during the period, the dealer makes a profit of Rs. 20000 but once the year is over, he has to bear a loss of Rs. 10000 as the new models enter the market. How many units of the product should he order for manufacturing so as to minimize the loss?