

Chapter 1

Elements of Decision Theory

Key words: Decisions, pay-off, regret, decision under uncertainty, decision under risk, expected value of perfect information, expected value of sample information, cost of irrationality, marginal analysis, sequential decision making, normal distribution.

Suggested readings:

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1.1 Introduction

A decision is an action, to be selected and taken by a decision-maker according to some pre-specified rule or strategy, out of several available alternatives, to facilitate the future course of action.

As a human being and a social element, we have to take several decisions in our every-day life; some of which are taken at random (e.g., which dress to wear today) and some other have a sound scientific basis (e.g., which course to enroll into). In decision analysis, we deal with the second type of decisions.

We give below some examples of decision-making problems:

- (i) In a multiple-choice question examination, a student gets 2 marks for each correct answer and loses half marks for each wrong answer whereas an unanswered question neither causes a gain nor a loss. Then depending upon his knowledge of the subject, he has to choose alternatives, which will maximize his score.
- (ii) A fresh fruits vendor sells on average 50 kg of grapes, with a standard deviation of 3 Kgs. Fruits sold on the same day yield him a profit of Rs. 20 per kg whereas stale fruit yield him a loss of Rs. 15 per kg. Then his problem is to determine the quantity of fruit, which will maximize his daily profit.

A decision, in general, results in a consequence which depends on two factors: (i) The decision chosen by the decision maker; and (ii) the actual state of the world (uncontrollable factors).

1.2 Elements of decision- making (problem)

Irrespective of decision-making problem, there are some elements, which are common to all the problems.

- (i) **An objective to be reached** The objective depends upon the type of the problem regarding which a decision is to be made, e.g., the ideal inventory level, reduction of the down-time of a machinery or maximization of the profit.
- (ii) **Courses of action** These are the alternative available from which the decision is to be made. These courses of action, also known as actions, acts or strategies, are under the control of the decision-maker.
- (iii) **State of nature** Also known as events, these are the results or consequences of the decision. These consequences are dependent upon certain factors, which are beyond the control of the decision-maker.

- (iv) **Uncertainty** This is the indefiniteness regarding the occurrence of event or outcome. Uncertainty arises due to uncontrollable factors associated with the states of nature.
- (v) **Pay-off** Also known as conditional profit or conditional economic consequence, a Pay-off is a calculable measure of the benefit or worth of a course of action and it represents the net benefit accruing from various combinations of alternatives and events.

A pay-off can be positive, zero or negative.

The conditional profits associated with a problem can be represented as a table or matrix, known as a pay-off matrix:

Table 1.1: Conditional pay-off matrix

<i>States of Nature</i>	<i>Courses of action</i>				
	<i>A1</i>	<i>A2</i>	<i>...</i>	<i>Aj</i>	<i>Am</i>
<i>E1</i>	<i>a11</i>	<i>a12</i>	<i>...</i>	<i>a1j</i>	<i>a1m</i>
<i>E2</i>	<i>a21</i>	<i>a22</i>	<i>...</i>	<i>a2j</i>	<i>a2m</i>
<i>...</i>					
<i>Ei</i>	<i>ai1</i>	<i>ai2</i>	<i>...</i>	<i>aij</i>	<i>aim</i>
<i>En</i>	<i>an1</i>	<i>an2</i>	<i>...</i>	<i>anj</i>	<i>anm</i>

In this matrix, various alternatives are shown along columns and the events are represented along the rows. Then the $(i, j)^{th}$ element a_{ij} of the table is the conditional profit associated with the i^{th} event and the j^{th} alternative.

An alternative way of representing the pay-offs is the tree diagram where the first bunch of branches represent the actions taken and the second fork represents the pay-offs associated with them. The above pay-off matrix in the tree form can be represented as follows:

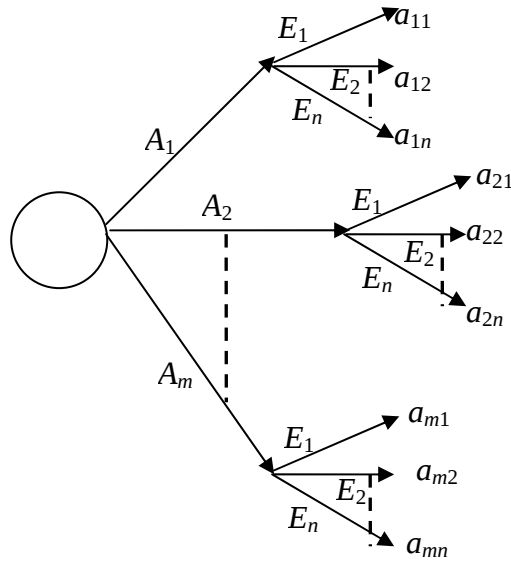


Fig. 1.1

Regret or opportunity loss table An opportunity loss is the loss occurring due to failure of not adopting that course of action, which would maximize the profit. It is the difference between the maximum pay-off and the pay-off of the action selected.

A loss cannot be negative. At most it can be zero.

If for the event E_i , M_i is the maximum pay-off then the regret table can be constructed as follows:

Table 1.2: Conditional opportunity loss matrix

States of Nature	Courses of action					
	A_1	A_2	...	A_j	...	A_m
E_1	$M_1 - a_{11}$	$M_1 - a_{12}$	...	$M_1 - a_{1j}$	...	$M_1 - a_{1m}$
E_2	$M_2 - a_{21}$	$M_2 - a_{22}$	...	$M_2 - a_{2j}$	...	$M_2 - a_{2m}$
...						
E_i	$M_i - a_{i1}$	$M_i - a_{i2}$	...	$M_i - a_{ij}$	...	$M_i - a_{im}$
E_n	$M_n - a_{n1}$	$M_n - a_{n2}$	...	$M_n - a_{nj}$	...	$M_n - a_{nm}$

The decision environment

Depending upon the information available, the decision environment may be one of the following types:

- (a) **Decisions with certainty:** In this environment, there is only one outcome of a decision. Linear programming problems and transportation problems fall under this environment.
- (b) **Decisions under conflict:** This environment deals with those situations when the states of nature are neither completely known nor completely unknown. The competitors marketing the same product deal with this environment.
- (c) **Decisions under uncertainty:** In this environment, any single decision may result in more than one type of outcomes, out of which the optimal one is to be selected.
- (d) **Decision under risk:** This environment is similar to the 'under uncertainty' environment except for the fact that the probabilities of occurrences of the outcomes can be stated from the past data.

Consider the following situations:

An individual is willing to invest Rs. 1, 00,000 in the stock market for one year period. Now, every one knows the uncertainty associated with the stock market. To make his investment as safe as possible, he has zeroed down three companies, say, A, B and C where he can make investment into. The current market price of the shares of all the three companies is Rs. 500 per share. As such he can purchase 200 shares which may belong to any of the three companies or may belong to a combination of any two or all the three companies. Then the problem is: how should he invest his money as to maximize his profit (or to minimize his loss in the worst situation)?

The following scenarios are possible:

- (i) He knows that at the end of one year, the prices of the stocks of the three companies would be Rs. 750, Rs. 500 and Rs. 400 respectively. Then each share bought would fetch him a profit of Rs. 250, Rs. 0 or -Rs.100 respectively. Obviously he will purchase the stock of company A as this investment would yield him maximum profit.

The decision is **decision under certainty**. He can obtain the optimal solution by an application of linear programming technique.

- (ii) If all the decisions could be made with certainty, we would have been a much happier lot. But as the experience tells us the situation is not so simple in general. Nobody can make a definite statement about the stock prices one year in advance.

Suppose that he knows that if the things remain more or less the same, the stocks of the three companies will grow, that of A will grow the fastest; that of B and C will grow more or less the same. If some thing unpredictable or a disaster occurs, the stock of company will be brought to earth; that of B and C will move upwards and would compete with each other. As

an example due to political conflicts across the world (oil crisis) accompanied by other factors (a fast growing economy of India) have resulted in a strong rupee. As a result infrastructure companies are booming whereas IT industries are in crisis. If he knows that the first situation will prevail, he would invest in company A, but in second situation he would like to invest in B or C. But nobody can predict unpredictable so what should he do?

This is ***decision making under conflict***.

- (iii) In general, the stock prices are not functions of unpredictable factors alone and there are more than two states of nature (disaster or no disaster). As a simple case, suppose that there are three states of nature, say, I, II and III. His presumption is that under different states of nature the stock of three companies will behave differently. Suppose the estimates of their stocks in three states of nature are given in the following table:

Table 1.3

States of Nature	Companies		
	A	B	C
I	900	470	500
II	300	850	930
III	550	1020	480

In case state of nature I prevails, he would have maximum benefit if he invest in company A, would earn nothing he if invest in company C but would be at loss if he invest in company B. In case state of nature II occurs, he would earn a loss as a result of his investment in company A, in investment in company B, he would again be at profit as in case of his investment in company C. In case of state of nature III, investment in A would yield him a little profit and investment in C is leading him to loss. But if he has invested in B, he would earn a big profit.

But again, he does not know whether state I or II or III of nature would prevail one year from now. So what should he do?

In this case, he is making a ***decision under uncertainty***.

- (iv) It is not that the different states of nature are equally likely to occur. For example, state I could be present conditions prevailing at that time; state II could be some economic reforms introduced and state III could be a political change over. In an election year, state III has highest probability of occurrence followed by state II and then state I. But in the regime of a

forward looking leadership, state II has highest probability of occurrence. Then the person can assign probabilities to different states of nature.

Let he assigns probabilities $\left(\frac{1}{3}, \frac{1}{4}, \frac{7}{12}\right)$ to the three states and now he estimates his expected profit (returns) on the basis of these probabilities.

This is **decision making under risk**.

Table 1.4

900	470	500
300	850	930
550	1020	480

In tree form

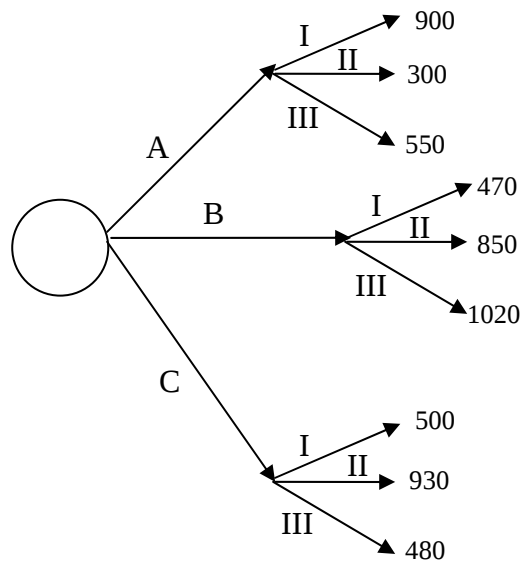


Fig. 1.2

We shall consider the last two categories of the decision environment as, in general, decisions are made in these two environments.

1.3 Decisions under uncertainty- Non probabilistic criteria

In this environment, only pay-offs are known. However, the likelihood of the events is completely unknown. A good decision is made by using all the available information to reach the objective set by the decision maker, although it may not result in a good outcome.

Several criteria or decision rules have been suggested to deal with such situations:

(i) **The maximin criterion** This criteria, practiced by the pessimistic decision makers, is based upon the 'conservative approach' to assume that the worst is going to happen. For each strategy, minimum pay-off is calculated and then among these minimum pay-offs, the strategy with the maximum pay-off is selected. The idea is to maximize the minimum gain.

The strategy is appropriate only when the conditional pay-offs are in terms of gains.

Example 1: A person wants to invest in one of the three investments plans: stock, bonds, or a saving account. It is assumed that the person is wishing to invest in one plan only. The conditional pay-offs of the investments are based on three potential economic conditions: high, normal or slow growth of the economy. The pay-off matrix is given by

Table 1.5: Conditional pay-off matrix

Investment Alternatives	Growth of Economy		
	High	Normal	Slow
Stock	Rs.10, 000	Rs. 6,500	-Rs. 4,000
Bonds	8, 000	6, 000	1, 000
Savings	5, 000	5, 000	5, 000

Determine the best investment plan according to the maximin criterion.

Sol:

Table 1.6

Investment	Minimum pay-off (Rs.)
Stock	- 4,000
Bonds	1,000
Savings	5,000

The maximum of these minimum pay-offs is Rs.5, 000 that corresponds to the third option. Hence the person should consider investing in savings account.

Example 2: A company, wishing to undertake a new marketing plan, has three alternatives:

- Introducing a new product with a new packing to replace the existing product at a very high price P_1 .
- A moderate change in the composition of the exiting product with a new packing at a moderately increased price P_2 .

- (c) A very small change in the composition of the existing product with a new packing at a slightly high price P_3 .

The three possible states of nature are

- (i) A high increase in sales n_1 .
(ii) No change in sales n_2 ; and
(iii) Decrease in sales n_3 .

The following table gives the pay-offs in terms of yearly profit from each of the strategy:

Table 1.7: Conditional pay-off matrix

Strategies	States of nature		
	n_1	n_2	n_3
P_1	Rs.7, 000	Rs. 3,500	Rs. 1,50
P_2	5, 000	4, 000	0
P_3	3, 000	3, 000	3, 000

Which strategy would be selected by a pessimistic decision maker?

Sol:

Table 1.8: Pay-off matrix

States of nature	Strategies		
	P_1	P_2	P_3
n_1	Rs.7, 000	Rs. 5,000	Rs. 3,000
n_2	3, 500	4, 000	3,000
n_3	1, 500	0	3, 000
Col minimum	1, 500	0	3, 000

A pessimistic decision maker would adopt the third strategy, i.e., a minimal change in the existing product is recommended.

- (ii) **The minimax criterion** This criterion is used when the decision is to be taken regarding costs. The costs are always minimized. The criterion suggests for determination of maximum possible cost for each alternative and then choosing best (minimum) cost among these worst (maximum) costs. This approach is practiced by conservative decision makers when the pay-offs are in terms of costs or losses.

Example 3: An HR manager has been assigned the job of making new recruitment for a new business assignment of a firm. The alternatives available before him are

- (i) Recruitment of unskilled labour which will then be trained R_1 ;
- (ii) Recruitment of semi-skilled workers R_2 ;
- (iii) Recruitment of trained workers R_3 ; and
- (iv) Outsourcing the job R_4 .

The four possible states of nature are

- (a) Decrease in profits P_1 ;
- (b) No increase P_2 ;
- (c) Moderate increase P_3 ; and
- (d) Substantial increase P_4 .

The costs associated with the different options are as follows

Table 1.9: Cost of recruitment

States of nature	Strategies			
	R_1	R_2	R_3	R_4
P_1	1	2	4	6
P_2	3	5	6	8
P_3	8	4	6	3
P_4	5	7	3	5

Find the best alternative using the minimax criterion.

Sol: The following table gives the worst costs associated with an option

Table 1.10

Alternatives	Maximum cost
R_1	8
R_2	7
R_3	6
R_4	8

The manager should go for the recruitment of trained workers.

(iii) **The maximax criterion** Practiced by the optimistic decision makers, this criterion calls for the selection of that strategy which corresponds to the highest pay-off among all the maximum pay-offs. The idea is to maximize the maximum gain.

Example 4: In example 1, determine the best investment plan according to the maximax criterion.

Sol:

Table 1.11

Investment	Maximum pay-off
Stock	10,000
Bonds	8,000
Savings	5,000

The highest pay-off among the maximum pay-offs is Rs. 10,000. The corresponding investment option, i.e., stocks should be selected by an optimistic investor.

Example 5: The three hot areas of technology development are IT, telecommunications, and biotechnology. The business environment may represent high, moderate or low growth. The expected rates of returns have been estimated according to the following table:

Table 1.12: Expected rate of return (%)

Growth	Business		
	IT	Telecommunications	Biotechnology
High	6.0	5.5	4.3
Moderate	3.2	2.7	2.5
Low	0.8	2.0	2.3

Determine the best business strategy for an optimistic investor.

Sol:

Table 1.13

Business	Maximum return (%)
IT	6.0
Telecommunications	5.5
Biotechnology	4.3

The best strategy for an optimistic investor is to invest in IT

(iv) **The minimin criterion** Again, this strategy is practiced by an optimistic investor and it calls for minimization of the minimum costs. The minimization of the minimum cost is equivalent to the maximization of the maximum profit.

Example 6: In example 3, which decisions the HR manager should take if he opts for minimin criterion?

Sol:

Table 1.14

Alternatives	Minimum cost
R_1	1
R_2	2
R_3	3
R_4	3

The manager should go for the recruitment of untrained workers if he opts for minimin criterion.

(v) **The savage (minimax regret) criterion** Consider the following situation:

Table 1.15: Expected rate of return (%)

Growth	Investments		
	Stocks	Bonds	Savings
High	20	15	14
Moderate	12	10	12
Low	8	9	10

An optimistic investor would always look for investment in stocks, whereas a pessimistic investor would always opt for savings. However, both the decisions are not good unless the economy is observing very high or very low rates of growth respectively.

An alternate decision strategy could be to minimize the maximum regret.

Table 1.16: Regret matrix

Growth	Investments		
	Stocks	Bonds	Savings
High	0	0	0
Moderate	8	5	2
Low	12	6	4
Maximum regret	12	6	4

According to this criterion, savings should be opted.

In case of cost matrix, we subtract the least cost from the other costs associated with that state of nature and the option, for which the maximum regret is minimum, is selected.

Example 7: The ABC Company has to make a decision from four alternatives relating to investments in a capital expansion programme. The different market conditions are the states of nature. The rates of return are as follows

Table 1.17: Expected rate of return (%)

Decisions	States of nature		
	θ_1	θ_2	θ_3
D_1	17	15	8
D_2	18	16	9
D_3	21	14	9
D_4	19	12	10

If the company has no information regarding the probability of occurrence of the three states of nature, recommend the best decision according to the savage principle.

Sol:

Table 1.18: Opportunity loss table

Strategies	States of nature			
	θ_1	θ_2	θ_3	Maximum opportunity loss
D_1	21-17=4	1	2	4
D_2	3	0	1	3
D_3	0	2	1	2
D_4	2	4	0	4

To minimize the maximum regret, the strategy D_3 should be opted.

Example 8: In example 3, find the best option using the savage criterion.

Sol:

Table 1.19: Regret matrix

States of nature	Strategies			
	R_1	R_2	R_3	R_4
P_1	0	1	3	5
P_2	0	2	3	5
P_3	5	1	3	0
P_4	2	4	0	2
Maximum regret	5	4	3	5

Alternative 3, i.e. trained workers should be recruited.

(vi) **The criterion of realism – Hurwicz criterion** In reality, a decision maker may neither be completely optimistic nor completely pessimistic but somewhere between the two extreme situations. The criterion of realism provides a mechanism of striking a balance between the two extreme situations by weighing them with certain degrees of optimism and pessimism.

The criterion calls for choosing a certain degree α of optimism ($0 \leq \alpha \leq 1$) so that $1-\alpha$ is the degree of pessimism. When $\alpha = 0$, it signifies complete pessimism and when $\alpha = 1$, it signifies complete optimism.

For each alternative, the Hurwicz factor H is, then, calculated as

$$H = \alpha \times \text{maximum pay-off} + (1-\alpha) \times \text{minimum pay-off}$$

The rule is to choose the strategy with the largest H .

Example 9: In example 1, find the best option using the criterion for realism if $\alpha = 0.6$.

Sol:

Table 1.20

Strategy	Maximum pay-off (Rs.)	Minimum pay-off (Rs.)	H
Stocks	10,000	-4,000	4,400
Bonds	8,000	1,000	5,200
Savings	5,000	5,000	5,000

Hurwicz criterion suggests bonds as the best option.

Example 10: A farmer wants to decide which of the three crops should he plant on his field. The produce depends upon the climate situation during the harvest period, which can be excellent, normal or bad. His estimated profits for each state of nature are given in the following table:

Table 1.21: Expected conditional profit

Climate conditions	Crops		
	A	B	C
Excellent	8000	3500	5000
Normal	4500	4500	5000
Bad	2000	5000	4000

If the farmer wants to sow only one crop, which one should he select if $\alpha = 0.7$.

Sol:

Table 1.22

Crop	Maximum pay-off (Rs.)	Minimum pay-off (Rs.)	H
A	8000	3500	6150
B	5000	4500	5050
C	5000	2,000	4100

According to Hurwicz criterion, crop A should be sown.

(vii) **The Laplace criterion** This criterion calls for making use of all the available information by assigning equal probabilities to every possible pay-off for each action and then selecting that alternative which corresponds to the maximum expected pay-off. If the pay-offs are in terms of costs, then the strategy with the least expected pay-off is selected.

Example 11: Find the best option of investment in example1 by Laplace criterion.

Sol: We assign equal probabilities to all the possible payoffs for each investment.

Table 1.23: Assignment of probabilities

Investment Alternatives	Growth of Economy		
	High	Normal	Slow
Stock	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Bonds	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Savings	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 1.24: Expected pay-off

Investment Alternatives	Growth of Economy			
	High	Normal	Slow	Expected pay-off
Stock	Rs.10, 000	Rs. 6,500	-Rs. 4,000	4167
Bonds	8, 000	6, 000	1, 000	5000
Savings	5, 000	5, 000	5, 000	5000

Using Laplace criterion, money can either be invested in bonds or in savings.

Example 12: In example 7, find the best option using Laplace criterion.

Sol:

Table 1.25: Expected rate of return (%)

Strategies	States of nature			
	θ_1	θ_2	θ_3	Expected pay-off
D_1	17	15	8	13.3
D_2	18	16	9	14.3
D_3	21	14	9	14.6
D_4	19	12	10	14.3

The best option using Laplace criterion is option 3, i.e. D_3 .

1.4 Decision making under risk

When we are making decision under uncertainty, we are working under the perception that the events are affected by the decisions that we make. But in reality this is not the situation. The occurrence of an event is not affected by the decisions that we make or the action that we perform. For example, in our investment problem, our choice of investment will not cause the economy to grow at high, normal or slow speed. Thus a decision should be taken which will maximize the benefits in the long run subject to the neutral occurrence of the events.

Expected value and expected pay-off

Choosing a decision with the largest expected value or pay-off is a strategy, which will maximize the benefits in the long run. Each pay-off is assigned a probability which may be chosen subjectively depending upon the decision maker or may be calculated from the past data or experience. Then the expected value of an action is the weighted sum of the conditional pay-offs, the weights being the corresponding probabilities.

Example 13: Consider the case of a baker who bakes and sells fresh cakes, which are demanded highly in the market. Because of the perishable nature of the product, the unsold cakes at the end of the day do not fetch him anything. On the basis of his past experience, the baker has estimated the following sales schedule:

Table 1.26

Event (Demand)	Probability of occurrence
20	0.05
21	0.15
22	0.30
23	0.25
24	0.15
25	0.10
	1.00

Any demand less than 20 units or more than 25 units is so rare that the probability of its occurrence is almost zero. Each unit of cake costs him Rs. 40 and he charges Rs. 70 for it so that his profit per unit is Rs. 30. If the demand is more than what he has baked, it is not possible to meet the demand on the same day and the demand is lost. Any unsold cake is a waste. Then the baker wants to know how many units he should bake in order to maximize his profit in the long run.

Sol: If D denotes the demand for cakes and S stands for the supply then the conditional pay-off function for the baker is given by

$$\text{Conditional pay-off} = \begin{cases} 70D - 40S & \text{if } S > D \\ 30S & \text{if } S \leq D \end{cases}$$

The conditional pay-offs have been calculated in the following table

Table 1.27: Conditional pay-off

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	600	560	520	480	440	400
21	0.15	600	630	590	550	510	470
22	0.30	600	630	660	620	580	540
23	0.25	600	630	660	690	650	610
24	0.15	600	630	660	690	720	680
25	0.10	600	630	660	690	720	750
	1.00						

We will now obtain the expected pay-off of each possible decision, which is the sum of the products of each conditional outcome and its probability.

Table 1.28: Expected pay-off

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	30	28	26	24	22	20
21	0.15	90	94.5	88.5	82.5	76.5	70.5
22	0.30	180	189	198	186	174	162
23	0.25	150	157.5	165	172.5	162.5	152.5
24	0.15	90	94.5	99	103.5	108	102
25	0.10	60	63	66	69	72	75
	1.00	600	626.5	642.5	637.7	615	582

If he bakes 22 cakes per day, it would give him an expected daily pay-off of Rs. 642.5. For any other number of cakes, his expected profit will be lower.

It should be noted that no other number of cakes would provide him a larger pay-off in the long run than 22 cakes per day. However, for some trials the pay-off may be higher (e.g., when $D = S > 22$). But such a strategy is sub optimal over a prolonged period of time. In fact 22 units are demanded just 30 percent of time. For 70 percent of time, the demand is different from 22 units. Still the decision to bake 22 units is giving him the largest expected pay-off.

An alternate approach- Expected loss

Loss analysis pertains to the losses incurred due to not adopting the optimal strategy. As we shall see, the loss analysis leads to the same decision as the expected profit analysis.

In our case, the baker suffers a loss of Rs. 40 on every unsold unit of cake if he bakes more cakes than demanded. In case his supply falls short of the demand, the result is a cash loss of Rs. 30 per unit besides the opportunity loss. Thus the conditional loss function of the baker is

$$\text{Conditional loss} = \begin{cases} 40(S - D) & \text{if } S \geq D \\ 30(D - S) & \text{if } S < D \end{cases}$$

i.e., the two components of the loss are the opportunity loss and the cash loss.

The conditional loss table is then obtained as follows:

Table 1.29: Conditional loss table

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	0	40	80	120	160	200
21	0.15	30	0	40	80	120	160
22	0.30	60	30	0	40	80	120
23	0.25	90	60	30	0	40	80
24	0.15	120	90	60	30	0	40
25	0.10	150	120	90	60	30	0

The expected loss table, is, then given by

Table 1.30: Expected loss

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	0	2	4	6	8	10
21	0.15	4.5	0	6	12	18	27
22	0.30	18	9	0	12	24	36
23	0.25	22.5	15	7.5	0	10	20
24	0.15	18	13.5	9	4.5	0	6
25	0.10	15	12	9	6	3	0
		78	51.5	35.5	40.5	63	99

Loss analysis also suggests baking 22 cakes per day.

Note: It may be noted that on adding the respective elements of conditional pay-off and conditional loss tables, we get the maximum pay-off associated with that event, i.e., the conditional loss is the difference between the best pay-off and the pay-off associated with that decision with respect to which conditional loss is being calculated.

Expected value of perfect information (EVPI)

When the baker bakes 22 units of cake per day, he is realizing, on average, a daily profit of Rs. 642.5 and his expected daily losses are Rs.35.5. This loss is occurring due to the fact that he is not having the advance information of the demand. Thus the expected loss is the cost of uncertainty in demand, and with the given extent of information, this cost is an irreducible cost.

If the baker had the perfect information about how many cakes would be demanded every day, he would have baked only that number of cakes so that he would neither fall short of supply nor would have been left with any unsold cakes at the end of that day.

Thus in presence of perfect information, his expected profit would have been given as in the following table:

Table 1.31

Demand	Probability of an event	Conditional pay-off	Expected pay-off
20	0.05	600	30
21	0.15	630	94.5
22	0.30	660	198
23	0.25	690	172.5
24	0.15	720	108
25	0.10	750	75
			678

Thus if he had baked only 20 cakes when the demand was going to be of 20 cakes only, 21 cakes when the demand was going to be of 21 cakes only and so on, his expected daily profit would have been Rs. 678. But in absence of this perfect information, his expected daily profit is only Rs. 642.5. The difference between the two amounts is the expected value of perfect information (EVPI), i.e., the expected loss associated with the optimal strategy in absence of perfect information.

This is the maximum amount that the baker can pay in order to obtain the complete information about the daily demand.

The EVPI also provides a measure of the additional sampling units. If the cost of sampling a unit is more than EVPI, additional sampling is not recommended.

Cost of irrationality This is the difference between the cost of uncertainty and the expected daily loss due to a sub optimal strategy, e.g., if the baker chooses to bake 23 cakes per day, he is incurring daily-expected loss of Rs.40.5. Then the cost of irrationality is Rs. 40.5 - Rs. 35.5 = Rs.5.

Items which have a salvage value

Now, suppose that the unsold cakes at the end of the day are not just thrown away but can be sold at next day also, albeit at a reduced price, i.e., the cakes have a salvage value. This, in fact, is the situation with most of the products and most of the products have a salvage value. If a product has a salvage value, it must be considered in calculating the pay-offs associated with the product.

Suppose that on the second day, the cakes can be sold for Rs.30 per unit.

Then the conditional loss on every unsold unit reduces by Rs. 30 and the conditional profit table is now given as

Table 1.32: Conditional pay-off when cakes have a salvage value

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	600	590	580	570	560	550
21	0.15	600	630	620	610	600	590
22	0.30	600	630	660	650	640	630
23	0.25	600	630	660	690	680	670
24	0.15	600	630	660	690	720	710
25	0.10	600	630	660	690	720	750

For example, in case of 21 units supplied and 20 units demanded, the conditional pay-off can be calculated as follows:

$$\begin{aligned}
 \text{Conditional pay-off} &= \text{profit of 20 units sold} - \text{cost of 21}^{\text{st}} \text{ unit} + \text{salvage value of 21}^{\text{st}} \text{ unit} \\
 &= \text{Rs.}(600 - 40 + 30) \\
 &= \text{Rs. 590}
 \end{aligned}$$

Now, we compute the expected profit.

Table 1.33: Expected pay-off

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	30	29.5	29	28.5	28	27.5
21	0.15	90	94.5	93	91.5	90	88.5
22	0.30	180	189	198	195	192	189
23	0.25	150	157.5	165	172.5	170	167.5
24	0.15	90	94.5	99	103.5	108	106.5
25	0.10	60	63	66	69	72	75
		600	628	650	660	660	654

With the given salvage value of the cake, decision to bake 23 units per day is the optimum decision.

The optimal strategy has changed due to the fact that conditional profits have been increased by the salvage value of the cake and the expected losses are reduced. The next best strategies are 24 or 25 units of cake.

Now, consider the situation when the salvage value of the cake is Rs. 15 per unit. In that situation, we have the following conditional and expected pay-off tables

Table 1.34: Conditional pay-off

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	600	575	550	525	500	475
21	0.15	600	630	605	580	555	530
22	0.30	600	630	660	635	610	585
23	0.25	600	630	660	690	665	640
24	0.15	600	630	660	690	720	695
25	0.10	600	630	660	690	720	750

Table 1.35: Expected pay-off

Event (Demand)	Probability of an event	Decision (Cakes baked)					
		20	21	22	23	24	25
20	0.05	30	28.75	27.5	26.25	25	23.75
21	0.15	90	94.5	96.75	87	83.25	79.5
22	0.30	180	189	198	190.5	183	175.5
23	0.25	150	157.5	165	172.5	166.25	160
24	0.15	90	94.5	99	103	108	104.25
25	0.10	60	63	66	69	72	75
		600	627.25	646.25	648.25	637.5	618

In this situation, although the best strategy is again to bake 23 cakes per day but the next best strategy is, now, to bake 22 units. A higher salvage value would lead to decision of baking more cakes whereas a lower salvage value would lead to decision of baking fewer cakes. Thus the optimal strategy depends upon the extent to which the expected losses can be covered by the salvage value of the product.

1.5 When the product has more than one salvage value

Marginal analysis Some times a product may have more than one salvage value. For instance, suppose that the shelf life of the cake is 3 days but on third day, it can be sold for Rs. 12 per unit only. Then in order to arrive at the optimal decision, several calculations are to be made. In such situations, we make use of marginal analysis and critical ratios to arrive at the optimal solution.

Suppose that the unit cost of the under stocking or overstocking remains constant, irrespective of the extent of under stocking or overstocking.

Marginal loss The loss of stocking an additional unit that could not be sold is called the marginal loss

Marginal profit The profit made due to sell of an additional unit is called the marginal profit.

Now, suppose that initially n units are supplied. If the supply is increased to $n + 1$ units, the additional unit is sold only if the demand is at least equal to $n + 1$ units. If the demand is less than or equal to n , the acquisition of the additional unit will result in a loss. If the marginal profit of selling an additional unit is denoted by MP and the marginal loss of an unsold unit be denoted by ML , then the expected loss of under stocking a unit in the new supply schedule will be given by

$$MP P(D \geq n+1) = MP (1 - P(D < n+1))$$

where D is the random variable denoting the demand.

Similarly expected loss of overstocking will be

$$ML P(D < n+1)$$

Then the rule for stocking an additional unit can be stated as follows:

Stock an additional unit if the expected marginal profit of overstocking is less than the expected marginal profit of under stocking, i.e., if

$$ML P(D < n+1) \leq MP (1 - P(D < n+1))$$

$$\text{Or, } P(D < n+1) \leq \frac{MP}{MP + ML} \quad (1.1)$$

R.H.S. of (1.1) is known as the **critical ratio** (CR), which suggests that a larger number of units should be stocked if the value of CR is high.

Alternatively, let p be the probability of selling an additional unit. Then with probability $1-p$, it will not be sold. Then expected profit of selling an additional unit is $p MP$ and the expected loss of not selling it is $(1-p) ML$. Then the rule says that an additional unit is justified till the point when

$$p MP = (1-p) ML$$

$$\Rightarrow p = \frac{ML}{MP + ML} \quad (1.2)$$

(1.1) and (1.2) suggest that CR is equal to $1-p$.

We, now, try to solve the baker's problem using marginal analysis.

The cumulative probability distribution of the baker is

Table 1.36

Event (n)	Probability	Cumulative probability = $P(D \geq n)$
20	0.05	1.00
21	0.15	0.95
22	0.30	0.80
23	0.25	0.50
24	0.15	0.25
25	0.10	0.10

Thus p decreases as the level of sales increases. According to the decision rule, an additional unit should be stocked as long as the probability of selling it is more than p .

For the baker

$$MP = \text{Rs.}30 \quad (= \text{Rs.}70 - \text{Rs.}40)$$

$$ML = \text{Rs.}40 = k_v \quad (\text{Cost of an additional unit of cake})$$

$$\Rightarrow p = \frac{ML}{MP + ML} = \frac{40}{30 + 40} = \frac{40}{70} \approx 0.57$$

Thus in order to justify another unit of cake, the cumulative probability of selling it must at least be 0.57.

The 57th percentile of the cumulative probability distribution corresponds to $n = 22$. So the baker should bake 22 units of cake in order to realize the maximum profit.

If, for each event, we calculate $p MP$ and $(1-p) ML$, then we get the following table

Table 1.37

Event (<i>n</i>)	Cumulative probability	<i>p MP</i>	(1- <i>p</i>) <i>ML</i>
20	1.00	30	0
21	0.95	28.5	2
22	0.80	24	8
23	0.50	15	20
24	0.25	7.5	30
25	0.10	3	36

Now, consider the case when the baker can realize a salvage value on the unsold cake. In that case

$$MP = \text{Rs.}30$$

$$ML = \text{Rs.}10 \quad (= \text{Rs.}40 - \text{Rs.}30)$$

$$\Rightarrow p = \frac{ML}{MP + ML} = \frac{10}{30 + 10} = 4 \frac{10}{70} = 0.25$$

Then we have the following table

Table 1.38

Event (<i>n</i>)	Cumulative probability	<i>p MP</i>	(1- <i>p</i>) <i>ML</i>
20	1.00	30	0
21	0.95	28.5	0.5
22	0.80	24	2
23	0.50	15	5
24	0.25	7.5	7.5
25	0.10	3	9

1.6 Sequential decision-making

Sometimes the decisions may have to be taken in sequence. Suppose that a person X wants to start some business, say, to start a travel agency. He has two options before him:

- (a) To start with a fleet of 5 luxury cars; or
- (b) To start with one deluxe bus and two luxury cars.

Due to financial restraints, he can have only one of the options at the beginning. However, after six months, depending upon how he has run the business, he can opt for any one of the following

- (i) If he has opted for (a) initially, he proceeds with the same;
- (ii) If he has opted for (a) initially and he is running the business successfully, then he can extend his business and opt for (b) also;
- (iii) If he has opted for (b) initially, he proceeds with the same; and
- (iv) If he has opted for (b) initially and he is running the business successfully, then he can extend his business and opt for (a) also.

Due to the nature of demand, the probability of success of (a) is 0.4, while for (b), it is 0.7. Initial investment in both the options is Rs.20, 00,000, which can be financed at an EMI of Rs. 25,000. If the project undertaken fails, nothing will be returned since the projects are financed. Project (a), if successful, will yield a monthly income of Rs. 50,000 and project (b) will have a monthly income of Rs. 35,000. Then the businessman wants to determine the optimal strategy.

In this case, the businessman has following options before him

- (i) Do nothing (0,0);
- (ii) Accept (a) only (a, 0);
- (iii) Accept (b) only (b, 0);
- (iv) Accept (a) first, and if successful then accept (b) (a, b); and
- (v) Accept (b) first, and if successful then accept (a) (b, a).

The businessman has to choose one of these options subject to the following four states of nature

- (i) Both (a) and (b) are successful ($a \ b$);
- (ii) (a) is successful but (b) is a failure ($a \ \bar{b}$);
- (iii) (b) is successful but (a) is a failure ($\bar{a} \ b$); and
- (iv) Both (a) and (b) are failures ($\bar{a} \ \bar{b}$).

The following table gives the conditional pay-offs

Table 1.39: Conditional pay-off

Event	Probability of an event	Decision				
		(0, 0)	(a, 0)	(b, 0)	(a, b)	(b, a)
$(a \ b)$	0.28	0	50000	35000	85000	85000
$(a \ \bar{b})$	0.12	0	50000	-25000	25000	-25000
$(\bar{a} \ b)$	0.42	0	-25000	35000	-25000	10000
$(\bar{a} \ \bar{b})$	0.18	0	-25000	-25000	-25000	-25000

Where $P(AB) = P(A)P(B)$

The expected pay-offs are given in the following table

Table 1.40: Expected pay-off

Event	Probability of an event	Decision				
		(0,0)	(a, 0)	(b, 0)	(a, b)	(b, a)
$(a \ b)$	0.28	0	14000	9800	23800	23800
$(a \ \bar{b})$	0.12	0	6000	-3000	3000	-3000
$(\bar{a} \ b)$	0.42	0	-10500	14700	-10500	4200
$(\bar{a} \ \bar{b})$	0.18	0	-4500	-4500	-4500	-4500
		0	5000	17000	11800	20500

The optimal strategy is to start with (b) and then to go for (a).

If we carry out the loss analysis, we have the following results

Table 1.41: Conditional loss

Event	Probability of an event	Decision				
		(0,0)	(a, 0)	(b, 0)	(a, b)	(b, a)
$(a \ b)$	0.28	85000	35000	50000	0	0
$(a \ \bar{b})$	0.12	50000	0	75000	25000	75000
$(\bar{a} \ b)$	0.42	35000	60000	0	60000	25000
$(\bar{a} \ \bar{b})$	0.18	0	25000	25000	25000	25000

Table 1.42: Expected loss

Event	Probability of an event	Decision				
		(0,0)	(a, 0)	(0, b)	(a, b)	(b, a)
$(a \ b)$	0.28	10200	9800	14000	0	0
$(a \ \bar{b})$	0.12	6000	0	9000	3000	9000
$(\bar{a} \ b)$	0.42	14700	25200	0	25200	10500
$(\bar{a} \ \bar{b})$	0.18	0	4500	4500	4500	4500
		30900	39500	27500	32700	24000

1.7 Continuous random variable – Use of normal distribution

Till now, we have assumed that the demand is a discrete random variable taking distinctly identifiable values. But this may not be the situation always and we may need to approximate the demand schedule by a continuous random variable.

If the demand schedule displays some specific distribution, we may proceed with the same. If not, then a practical solution is the use of normal distribution.

We know that a normal distribution is always characterized by its mean (μ) and variance (σ^2). Also, we know that if a random variable $X \sim N(\mu, \sigma^2)$, then the random variable Z , defined as

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1).$$

Then the normal probability tables can be used to reach at the optimum decision.

Consider a salesman who sells some perishable items the unit cost of which is Rs. 200 and which can be sold for Rs. 450 per piece. Due to perishable nature of the item, if not sold on the same day, it is worth nothing. The salesman estimates that the sales are distributed normally with mean 50 and variance 225. He wants to determine the optimal number of items that should be purchased per day so that he is able to optimize his profit.

Using the marginal analysis, we know that the maximum probability p required to stock an additional unit is

$$p = \frac{ML}{ML + MP} = \frac{200}{200 + 250} = \frac{200}{450} \approx 0.44$$

So, if the salesman is sure that with probability 0.44, he would be able to sell an additional unit, he can stock it. Then, the job of the salesman is to find that point on the normal curve which corresponds to area = 0.44.

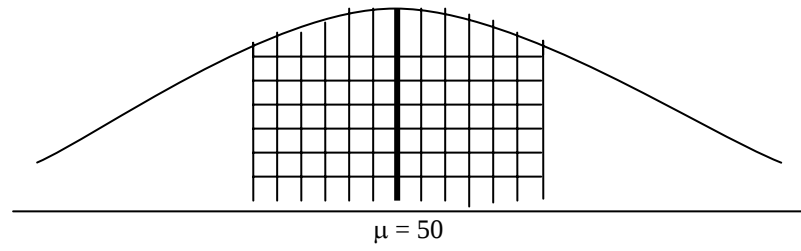


Fig. 1.3

Using the normal tables, we have $Z = 0.15$. Then,

$$0.15 = \frac{X - 50}{15}$$

$$\Rightarrow X = 50 + 2.25 = 52$$

This is the optimal order, which the salesman must put to optimize the profit.

Problems

1. A businessman has three strategies A, B and C, which according to the states of nature X, Y, Z and W may result in the following conditional pay-offs:

Table 1.43

States of nature	Strategy		
	A	B	C
X	4	-2	7
Y	0	6	3
Z	-5	9	2
W	3	-1	4

What should be the course of action according to

- (a) Maximin criterion?
 - (b) Minimax regret criterion?
 - (c) Maximum expected value if all the events have equal probability of occurrence?
2. Construct a conditional loss table from the above data.
 - (a) What are the cost of uncertainty and the expected value of perfect information?
 - (b) What is the cost of irrationality?

It is given that $P(X) = 0.3$; $P(Y) = 0.4$; $P(Z) = 0.2$ and $P(W) = 0.1$.
 3. A company has proposals for four alternative investment plans. Since these investments are to be made in future, the company foresees different market conditions as expressed in the form of states of nature. The following table summarizes the decision alternatives, the various states of nature and the rate of return associated with each state of nature:

Table 1.44

Decision alternatives	States of nature		
	A	B	C
X	17%	15%	8%
Y	18%	16%	9%
Z	21%	15%	9%
W	19%	12%	10%

If the company has no information regarding the probability of occurrence of these states of nature, recommend decisions according to the following decision criteria:

- (a) Maximax criterion;
- (b) Maximin criterion;
- (c) Minimax regret criterion;

- (d) Laplace criterion; and
- (e) Hurwicz alpha criterion ($\alpha = 0.75$).
4. An Informatics company summarizes international financial information reports (on a weekly basis), prints sophisticated data, and forecasts that are purchased weekly by financial institutions, banks, and insurance companies. The information is expensive and the demand for the information is limited to 30 reports per week. The possible demands are 0, 10, 20, and 30 units per week. The profit per report sold is Rs. 4,000. For each unsold report, the loss is Rs. 1500. No extra production is possible during a week. Further there is an additional penalty of Rs. 500 per report for not meeting the demand. Unsold reports cannot be carried over the next week. Find out the number of reports to be produced if
- (i) Maximin strategy is adopted; and
- (ii) Maximin strategy is adopted.
5. An investor is given the following investment options and the percentage rates of return

Table 1.45

Decision alternatives	States of nature (Market conditions)		
	Low	Medium	High
Bonds	7%	10%	15%
Equity	-10%	12%	25%
Real estate	12%	18%	30%

Over the past 300 days, 150 days have witnessed medium market conditions, and 60 days have witnessed high market conditions. Find the optimum investment strategy for the investment.

6. A child specialist purchases Hepatitis B vaccines on every Monday. Because of the nature of the vaccines, all the unused vials are to be discarded at the end of the week. The past data reveals the following information:

Table 1.46

Number of vials used per week	Frequency of the events
20	15
30	20
40	10
50	5

Using marginal analysis, determine the number of vials to be purchased per week if the doctor spends Rs. 250 per vials and charges Rs. 500 per patient. One vial is administered to a patient.

7. XYZ Corporation manufactures automobile spare parts and sells them in lots of 10,000 parts. The company has a policy of inspecting each lot before it is actually shipped to the retailers. The company has demarcated five inspection categories according to the percentage of defectives contained in each lot. The daily inspection chart for past 100 inspection reveals the following information

Table 1.47

Lot category	Proportion of defective items	Frequency
Excellent (A)	0.02	25
Good (B)	0.05	30
Acceptable (C)	0.10	20
Fair (D)	0.15	20
Poor (E)	0.20	5

The management is considering two possible courses of action

- (i) Shut down the entire plane operations and thoroughly inspect each machine;
- (ii) Continue production as it is now but offer the customer a refund for defective items that are discovered and subsequently returned.

The first alternative will cost Rs. 6,00,000 while the second alternative will cost company Rs. 10 for each defective item that is returned.

What is the optimum decision for the company? Find EVPI.

8. An engineering firm has installed a machine costing Rs. 4,00,000. The firm is in process of deciding on an appropriate number of a spare part required for repairs. The unit cost of the part is Rs. 4,000, and is available only if ordered now. In case the machine fails and no spare part is available, the cost of the company of mending the plant would be Rs. 20,000. The estimated failure schedule for the plant for eight years period is as follows:

Table 1.48

Failures during eight years period	Probability
0	0.1
1	0.2
2	0.3
3	0.2
4	0.1
5	0.1
6	0

Ignoring the time value of money, find

- (a) The optimal number of units of the part on the basis of
 - (i) Minimax principle;
 - (ii) Minimax principle;
 - (iii) Laplace criterion; and
 - (iv) Expected monetary value criterion.
- (b) The expected number of failures in the eight years period; and
- (c) EVPI.

9. ABC Engineering Co. is planning to increase its production capacity. It is considering two investment alternatives

- (i) Expansion of the plant at an estimated cost of Rs. 20,00,000; and
- (ii) Modernization of the existing plant at a cost of Rs. 8,00,000.

The company believes that over the pay back period, the demand will either be high or moderate. The respective probabilities of either of the events are 0.4 and 0.6. If the demand were high, expansion would yield additional revenue of Rs. 40,00,000 whereas modernization would yield additional Rs. 15,00,000. On the other hand, if the demand is moderate, then the additional yield for expansion would be Rs. 10,00,000 and for modernization, it would be Rs. 4,00,000.

Before actually deciding on whether to expand or to modernize, the management is considering to engage ALPHA consultants for performing an intensive marketing analysis and processing the data. Based on the analysis, ALPHA consultants will predict whether the demand will be high or moderate. The past experience shows that their prediction for high demand is correct 80% of time. For moderate demand their prediction are correct 70% of time. The cost of hiring ALPHA consultants is Rs. 40,000.

- (i) From the above information, determine the optimal decision that the company must take;
- (ii) Find whether it is advisable to engage ALPHA consultants and if so, will the optimal decision arrived at in part (i) change.

10. A farmer grows different types of flowers on his land. The most demanded flower is rose. He wants to decide the land to be allocated for the production of rose to maximize his profits. The cost of growing, packing, and marketing roses is Rs. 15,000 per acre. The produce can be sold for Rs. 25,000 per acre. The unsold flowers at the end of the day are worthless. The farmer has estimated the following demand distribution for roses:

Table 1.49

Acres	Probability	Acres	Probability
5	0.02	13	0.08
6	0.03	14	0.06
7	0.05	15	0.04
8	0.10	16	0.04
9	0.10	17	0.03
10	0.10	18	0.03
11	0.15	19	0.01
12	0.15	20	0.01

- (a) Use the critical ratio to find the number of acres where roses should be planted to maximize the expected value.
- (b) What is the expected cost of best decision?
- (c) If the unsold flowers can be used elsewhere for Rs. 2,000 per acre, what will be the decision?
- (d) For Rs. 8,000, a survey can be done to get a better estimate of the potential demand. Is the information worth obtaining for this cost?
- (e) If there is a loss of Rs. 1500 per acre for not meeting the demand, repeat parts (a) to (d).
11. Consider the following loss table and complete the pay-off table given below:

Table 1.50

States Actions	I	II	II
I	6	3	0
II	0	1	2
III	3	0	4

Table 1.51

States Actions	I	II	II
I	9	7	
II	12		
III			

Find the expected pay-offs and expected losses if

$$P(I) = 0.2; P(II) = 0.5; P(III) = 0.3;$$

On the basis of these two criteria, which action do you think is optimal?